Conferencing Encoders for Compound and Arbitrarily Varying Multiple-Access Channels

Holger Boche
Joint work with Moritz Wiese and Igor Bjelaković

Technische Universität München

Interactive Information Theory
January 17, 2012

Outline

Introduction

Compound MAC with Conferencing Encoders

Arbitrarily Varying MAC with Conferencing Encoders

Gains of Conferencing

Conclusion

Outline

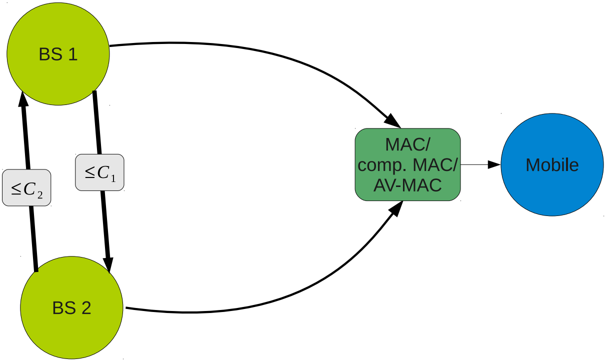
Motivation

- Power reduction and spectrum scarcity cause interference to be the main factor that limits performance of modern wireless systems.
- The potential of pure cellular concepts to deal with interference is close to being exhausted.
- Investigate potential of inter-cell cooperation [Karakayali, Foschini, Valenzuela 2006].
  - To be included in 5G wireless systems like LTE-Advanced.
- *Problem 1:* Full cooperation is too complex and uses too many resources
  - Partial cooperation needs to be investigated.
  - Partial channel state information (CSI) needs to be considered.
- *Problem 2:* How can one cope with interference from co-existing networks run by different providers?
  - Uncoordinated WLAN hot spots
  - Frequency co-sharing in 5G mobile networks

Conferencing Encoders

Willems introduced the concept of conferencing encoders in information theory [Willems 1982].

- models rate-limited cooperation between base stations.



Willems' Conferencing Protocol

- Consists of an interactive exchange of information about the messages m_1, m_2 present at encoder 1 and 2, resp.

An I-iterations conferencing protocol has the form

$$\begin{array}{rr}
\mathcal{M}_1 \xrightarrow{c_{1,1}} \mathcal{V}_{1,1} & \mathcal{M}_2 \xrightarrow{c_{2,1}} \mathcal{V}_{2,1} \\
\mathcal{M}_1 \times \mathcal{V}_{2,1} \xrightarrow{c_{1,2}} \mathcal{V}_{1,2} & \mathcal{M}_2 \times \mathcal{V}_{1,1} \xrightarrow{c_{2,2}} \mathcal{V}_{2,2} \\
\mathcal{M}_1 \times \mathcal{V}_{2,1} \times \mathcal{V}_{2,2} \xrightarrow{c_{1,3}} \mathcal{V}_{1,3} & \mathcal{M}_2 \times \mathcal{V}_{1,1} \times \mathcal{V}_{1,2} \xrightarrow{c_{2,3}} \mathcal{V}_{2,3} \\
\vdots & \vdots \\
\mathcal{M}_1 \times \mathcal{V}_{2,1} \times \ldots \times \mathcal{V}_{2,I-1} \xrightarrow{c_{1,I}} \mathcal{V}_{1,I} & \mathcal{M}_2 \times \mathcal{V}_{1,1} \times \ldots \times \mathcal{V}_{1,I-1} \xrightarrow{c_{2,I}} \mathcal{V}_{2,I}
\end{array}$$

Here

- $\mathcal{M}_\nu$ = message set of sender $\nu \in \{1, 2\}$,
- $\mathcal{V}_{\nu,i}$ finite set with $|\mathcal{V}_{\nu,i}| = V_{\nu,i}, \qquad \nu = 1, 2,\ i = 1, \ldots, I.$

MAC Coding with Rate-Constrained Conferencing

Given a multiple-access channel (MAC) with input alphabets $\mathcal{X}$ and $\mathcal{Y}$.

- Conferencing is part of coding.
- With conferencing capacities $C_1, C_2 \geq 0$, for a blocklength-n code, the above sets $\mathcal{V}_{1,1}, \ldots, \mathcal{V}_{2,I}$ need to satisfy

$$\frac{1}{n} \log V_{1,1} \cdots V_{1,I} \leq C_1, \qquad \frac{1}{n} \log V_{2,1} \cdots V_{2,I} \leq C_2.$$

- Describes the rate-constrained iterative exchange of information about the messages present at the encoders.
- The encoding functions have the form

$$f_1 : \mathcal{M}_1 \times \mathcal{V}_{2,1} \times \ldots \times \mathcal{V}_{2,I} \to \mathcal{X}^n,$$
$$f_2 : \mathcal{M}_2 \times \mathcal{V}_{1,1} \times \ldots \times \mathcal{V}_{1,I} \to \mathcal{Y}^n.$$

Channel Models

- The Compound MAC with conferencing encoders models channel state uncertainty in a downlink network with cooperating base stations.
- It is also the key to the solution of the coding theorem for AV-MACs.
- The Arbitrarily Varying MAC (AV-MAC) with conferencing encoders models a downlink network with cooperating base station suffering from interference from networks operating in the same band.
- New effects occur in AV-MACs – conferencing can change the whole channel structure.

Outline

The Compound MAC

The Compound MAC models channel state uncertainty.

- A Compound MAC with state set $\mathcal{S}$ is a family

$$\mathcal{W} = \{W_s : s \in \mathcal{S}\}$$

  of discrete memoryless MACs $W_s : \mathcal{X} \times \mathcal{Y} \to \mathcal{P}(\mathcal{Z})$.
- The transmission of words $\mathbf{x} \in \mathcal{X}^n$ and $\mathbf{y} \in \mathcal{Y}^n$ is governed by the probabilities

$$W_s^n(\mathbf{z}|\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{n} W_s(z_i|x_1, y_i) \qquad (s \in \mathcal{S}).$$

- The encoders and the decoder only have partial CSI.
- Partial CSI is modeled as partitions of the state set $\mathcal{S}$ for the encoders and the decoder.
- We restrict ourselves here to the case with no CSI at all.
  - arbitrary CSI treated in [W,B,B,Jungnickel 2011, TransIT]

Coding for the Compound MAC

- The conferencing codes are as described before.
- Through conferencing, the codewords of each encoder depend on both messages m_1 and m_2. They are called $\mathbf{x}_{m_1 m_2}$ and $\mathbf{y}_{m_1 m_2}$.
- The decoding sets are called $D_{m_1 m_2}$.
- The average error for the code is given by

$$\max_{s \in \mathcal{S}} \frac{1}{|\mathcal{M}_1||\mathcal{M}_2|} \sum_{m_1, m_2} W_s^n(D_{m_1 m_2}^c | \mathbf{x}_{m_1 m_2}, \mathbf{y}_{m_1 m_2}).$$

- To achieve a rate pair with conferencing capacities $C_1, C_2 \geq 0$, one may use codes whose conferencing protocol satisfies the rate restriction

$$\frac{1}{n} \log V_{1,1} \cdots V_{1,I} \leq C_1, \qquad \frac{1}{n} \log V_{2,1} \cdots V_{2,I} \leq C_2.$$

The Capacity Region

The capacity region of the Compound MAC with conferencing capacities $C_1, C_2 \geq 0$ equals the closure of

$$\bigcap_{U,X,Y} \bigcup_{s \in \mathcal{S}} \Big\{ (R_1, R_2) \in [0, \infty)^2 :$$
$$R_1 \leq I(Z_s; X|Y, U) + C_1,$$
$$R_2 \leq I(Z_s; Y|X, U) + C_2,$$
$$R_1 + R_2 \leq \min\big(I(Z_s; X, Y|U) + C_1 + C_2, I(Z_s; X, Y) \big) \Big\},$$

where $P_{Z_s|U,X,Y} = W_s$ and X, Y are independent given U.

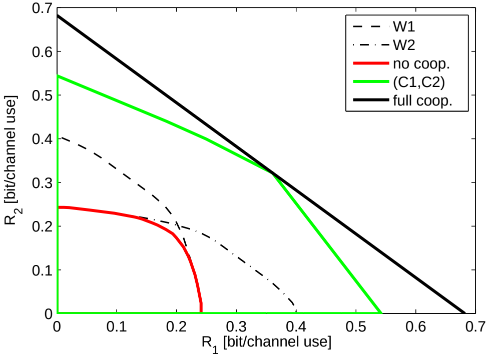
- The region is achieved using non-iterative conferencing protocols, i.e. with $I = 1$.

Numerical Example

Let $\mathcal{X} = \mathcal{Y} = \mathcal{Z} = \mathcal{S} = \{0,1\}$ and the Compound MAC $\{W_1, W_2\}$ with

$$W_1 = \frac{1}{10}\begin{pmatrix} 9 & 1 \\ 4 & 6 \\ 6 & 4 \\ 0 & 10 \end{pmatrix} \quad \text{and} \quad W_2 = \frac{1}{10}\begin{pmatrix} 9 & 1 \\ 6 & 4 \\ 4 & 6 \\ 0 & 10 \end{pmatrix},$$

where the output corresponding to the input combination (a,b) is written in row $2a+b+1$. With $C_1 = C_2 \approx 0.301$,



Proof of Achievability

Reduced to the Compound MAC with Common Message. Given a rate pair (R_1, R_2),

- Set

$$\mathcal{M}_1 = \mathcal{M}_{1,p} \times \mathcal{M}_{1,c}, \qquad \mathcal{M}_2 = \mathcal{M}_{2,p} \times \mathcal{M}_{2,c}$$

  with $\frac{1}{n}\log|\mathcal{M}_\nu| = 2^{nR_\nu}$ and $\frac{1}{n}\log|\mathcal{M}_{\nu,c}| = \min(R_\nu, C_\nu)$.
  - Uniform partitions of $\mathcal{M}_1$ and $\mathcal{M}_2$.
- Set $c_\nu(m_\nu) = c_\nu(m_{\nu,p}, m_{\nu,c}) = m_{\nu,c} \quad (\nu = 1, 2)$.
- The joint result of conferencing $(m_{1,c}, m_{2,c})$ is a uniformly distributed common message from $\mathcal{M}_{1,c} \times \mathcal{M}_{2,c}$.
- Use codes for the Compound MAC with Common Message for the message set $(\mathcal{M}_{1,c} \times \mathcal{M}_{2,c}) \times \mathcal{M}_{1,p} \times \mathcal{M}_{2,p}$.

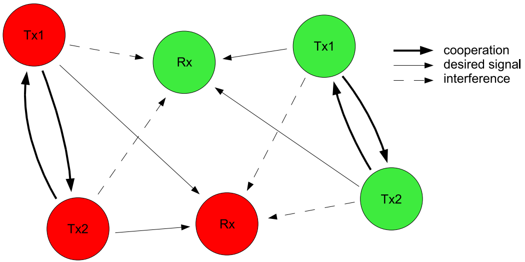
Weak and Strong Converse

- There is a weak converse.
- A strong converse for compound channels can only be shown for the maximal error [Ahlswede 1967].
- For MACs, the maximal error capacity region differs from the average error capacity region in general [Dueck 1978].
- For MACs, the capacity region is only known for the average error.

Outline

The Arbitrarily Varying MAC (1)

The AV-MAC models the interference of networks with a MAC they do not cooperate with and which operate in the same band.



For example:

- Uncoordinated WLAN hot spots
- Frequency co-sharing in 5G mobile networks

The Arbitrarily Varying MAC (2)

- The AV-MAC with state set $\mathcal{S}$ is also given by a family

$$\mathcal{W} = \{W_s : s \in \mathcal{S}\}.$$

- However, the states may change at every time instant, so the set of n-block transition probabilities consists of

$$W^n(\mathbf{z}|\mathbf{x}, \mathbf{y}|\mathbf{s}) = \prod_{i=1}^{n} W_{s_i}(z_i|x_i, y_i)$$

for $\mathbf{s} \in \mathcal{S}^n$.

Coding for the AV-MAC

- The conferencing codes are as described before.
- The average error of a conferencing code for transmission over the AV-MAC is given by

$$\sup_{\mathbf{s}\in\mathcal{S}^n} \frac{1}{|\mathcal{M}_1||\mathcal{M}_2|} \sum_{m_1,m_2} W^n(D^c_{m_1m_2}|\mathbf{x}_{m_1m_2},\mathbf{y}_{m_1m_2}|\mathbf{s}).$$

- To achieve a rate pair with conferencing capacities $C_1, C_2 \geq 0$, one may use codes whose conferencing protocol satisfies the rate restriction

$$\frac{1}{n}\log V_{1,1}\cdots V_{1,I} \leq C_1, \qquad \frac{1}{n}\log V_{2,1}\cdots V_{2,I} \leq C_2.$$

New Effects in Uncoordinated Networks

The capacity region of the AV-MAC exhibits a dichotomy. This is characterized by symmetrizability [Ericson 1985], [Gubner 1990].

The AV-MAC $\mathcal{W}$ is $(\mathcal{X},\mathcal{Y})$-symmetrizable if there is a stochastic matrix $\sigma : \mathcal{X} \times \mathcal{Y} \to \mathcal{P}(\mathcal{S})$ such that for all $x, x' \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$ and $z \in \mathcal{Z}$

$$\sum_{s\in\mathcal{S}} W(z|x,y)\sigma(s|x',y') = \sum_{s\in\mathcal{S}} W(z|x',y')\sigma(s|x,y).$$

$\mathcal{W}$ is $\mathcal{X}$-symmetrizable if there is a stochastic matrix $\sigma_1 : \mathcal{X} \to \mathcal{P}(\mathcal{S})$ such that for all $x, x' \in \mathcal{X}$ and $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$

$$\sum_{s\in\mathcal{S}} W(z|x,y)\sigma_1(s|x') = \sum_{s\in\mathcal{S}} W(z|x',y)\sigma_1(s|x).$$

$\mathcal{Y}$-symmetrizability is defined analogously.

Capacity Region of the AV-MAC

Theorem [W,B 2011]

The capacity region of $\mathcal{W}$ with conferencing capacities $C_1, C_2 > 0$ equals $\{(0,0)\}$ if and only if it is $(\mathcal{X},\mathcal{Y})$-symmetrizable. Otherwise it equals the capacity region of the compound MAC with conferencing capacities C_1, C_2 given by

$$\overline{\mathcal{W}} := \{W_q = \sum_{s\in\mathcal{S}} W_s q(s) : q \in \mathcal{P}(\mathcal{S})\}.$$

- Conferencing is as simple as for Compound MACs.

Proof: Robustification

- Turn good codes for the compound MAC with cooperating encoders into good random codes for the AV-MAC.

Random code: A finite family of codes $\{\mathbf{x}^\gamma_{m_1m_2}, \mathbf{y}^\gamma_{m_1m_2}, D^\gamma_{m_1m_2}\}$ together with a random variable Γ living on this family. The average error incurred by such a code equals

$$\max_{\mathbf{s}\in\mathcal{S}^n} \frac{1}{M_1M_2} \sum_{m_1,m_2} \sum_{\gamma} W^n((D^\gamma_{m_1m_2})^c | \mathbf{x}^\gamma_{m_1m_2}, \mathbf{y}^\gamma_{m_1m_2} | \mathbf{s}) P_\Gamma(\gamma).$$

[Ahlswede 1980]: Good deterministic codes for the compound MAC $\overline{\mathcal{W}}$ can be turned into good random codes for the AV-MAC $\mathcal{W}$ by randomizing over polynomially many permutations of the codewords and the decoding sets.

Proof: Elimination of Correlation

- Turn a good random code into a good deterministic code.

Idea: Use a random code with the desired rates. Prefix a deterministic code to it which specifies which random code will be used. [Ahlswede 1978]

Random code blocklength n, polynomially many deterministic codes constituting the random code

- prefixed code has blocklength $\log n$ and – asymptotically – arbitrarily small rate.

Thus if any positive rate is achievable deterministically, then this derandomization method can be used.

Proof: Achieving a small positive rate

- A small positive rate pair is achievable if $\mathcal{W}$ is not $(\mathcal{X}, \mathcal{Y})$-symmetrizable.

$0 < \tilde{R} < 2\min\{C_1, C_2\}$ is deterministically achievable by the single-user AVC $\mathcal{W}$ with input alphabet $\mathcal{X} \times \mathcal{Y}$ if $\mathcal{W}$ is not $(\mathcal{X}, \mathcal{Y})$-symmetrizable [Csiszár, Narayan 1988, capacity of single-user AVCs].

$|\mathcal{M}_1| = |\mathcal{M}_2| = 2^{n\tilde{R}/2} =: 2^{nR}$

- every encoder informs the other encoder completely about the message it would like to send
- The encoders use the codeword corresponding to the message pair from the single-user code achieving $\tilde{R}$.

Converse

The converse follows from the converse for the compound channel together with the result of [Csiszár, Narayan 1988].

- If $\mathcal{W}$ is $(\mathcal{X}, \mathcal{Y})$-symmetrizable, then by [Csiszár, Narayan 1988], every code incurs at least an error of $1/4$.
  - "Almost" a strong converse.
- If $\mathcal{W}$ is not $(\mathcal{X}, \mathcal{Y})$-symmetrizable, then at least the corresponding compound region is achievable. Clearly, this cannot be exceeded by the AV-MAC.
  - Weak converse.

Open problems for the AV-MAC

- Is the direct approach to the AVC capacity [Csiszár, Narayan 1988] feasible for the AV-MAC?
  - Conjecture: Impossible for the Quantum AVC
    $\rightsquigarrow$ Ahlswede's approach must be taken [Ahlswede,B,B,Nötzel 2010].
- The maximal error for arbitrarily varying channels leads to the unsolved problem of zero-error capacity.

For average errors –

- can we obtain a strong converse for non-zero capacity regions?
- still no complete description of the non-conferencing capacity region.

Outline

Gains for Compound MACs

Comparison of Compound MAC without encoder cooperation with Compound MAC with $C_1, C_2 > 0$.

- Gains, if existent, are continuous in C_1, C_2 including $C_1 = C_2 = 0$.
- If single-user (sum) capacity of Compound MAC $\mathcal{W} : \mathcal{X} \times \mathcal{Y} \to \mathcal{P}(\mathcal{Z})$ equals zero
  $\rightsquigarrow$ nothing is gained by conferencing.
- If single-user capacity greater then zero
  $\rightsquigarrow$ capacity region grows from non-cooperative to full-cooperation region
  $\rightsquigarrow$ linear in C_1, C_2 until cutoff.

Gains for AV-MACs: Background

Theorem [Ahlswede, Cai 1999]

The capacity region of $\mathcal{W}$ without encoder cooperation contains a pair (R, R) with $R > 0$ if $\mathcal{W}$ is neither $\mathcal{X}$- nor $\mathcal{Y}$- nor $(\mathcal{X}, \mathcal{Y})$-symmetrizable.

In that case

Theorem [Jahn 1981]

If the capacity region of $\mathcal{W}$ without encoder cooperation contains a pair (R, R) with $R > 0$, then it equals the capacity region of the compound channel $\overline{\mathcal{W}}$ without conferencing.

- If an AV-MAC is $(\mathcal{X}, \mathcal{Y})$-symmetrizable or both $\mathcal{X}$- and $\mathcal{Y}$-symmetrizable, then its capacity region equals $\{(0, 0)\}$.
- If it is either $\mathcal{X}$- or $\mathcal{Y}$-symmetrizable, then its capacity region is at most one-dimensional.

Gains for AV-MACs

Comparison of AV-MAC without encoder cooperation with Compound MAC with $C_1, C_2 > 0$.

- If $\mathcal{W}$ is $(\mathcal{X}, \mathcal{Y})$-symmetrizable, then conferencing does not help.
- If $\mathcal{W}$ is
  - not $(\mathcal{X}, \mathcal{Y})$-symmetrizable,
  - but $\mathcal{X}$- and $\mathcal{Y}$-symmetrizable,

  then

  - its capacity region without conferencing equals zero,
  - its capacity region with $C_1, C_2 > 0$ equals the capacity region of the compound MAC $\overline{\mathcal{W}}$ with C_1, C_2.

  $\rightsquigarrow$ discontinuous gains possible when enabling conferencing.
- Example: The AV-MAC with $\mathcal{X} = \mathcal{Y} = \mathcal{S} = \{0, 1\}$, $\mathcal{Z} = \{0, \dots, 3\}$,

$$z = x + y + s,$$

  is not $(\mathcal{X}, \mathcal{Y})$-symmetrizable but both $\mathcal{X}$- and $\mathcal{Y}$-symmetrizable.

Outline

Conclusion

- Base station cooperation is used in 5G networks to mitigate interference.
- Conferencing is an information-theoretic model of rate-limited base station cooperation.
- The Compound MAC with conferencing encoders models partial CSI and is the key to the AV-MAC with conferencing encoders.
- The AV-MAC models interference from the same band as occurring in frequency co-sharing.
- In every case, the optimal conferencing protocol is a simple non-iterative protocol.
- For AV-MACs, conferencing may enable discontinuous transmission gains.
- Still open problems regarding maximal error criterion and strong converses.

References

- R. Ahlswede, "Elimination of Correlation in Random Codes for Arbitrarily Varying Channels,", *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, no. 44, pp. 159–175, 1978.
- R. Ahlswede, "Coloring Hypergraphs: A New Approach to Multi-user Source Coding—II," *J. Comb. Inform. Syst. Sci.*, vol. 5, no. 3, pp. 220–268, 1980.
- R. Ahlswede, "Certain Results in coding theory for compound channels I," *Proc. Colloquium Inf. Th. Debrecen (Hungary)*, pp. 35–60, 1967.
- R. Ahlswede, I. Bjelaković, H. Boche, J. Nötzel, "Quantum capacity under adversarial quantum noise: arbitrarily varying quantum channels", submitted to *Commun. Math. Phys.*, 2010.
- R. Ahlswede and N. Cai, "Arbitrarily Varying Multiple-Access Channels Part I—Ericson's Symmetrizability Is Adequate, Gubner's Conjecture Is True,", *IEEE Trans. Inf. Theory*, vol. 45, no. 2, pp. 742–749, 1999.
- I. Csiszár and P. Narayan, "The Capacity of the Arbitrarily Varying Channel Revisited: Positivity, Constraints", *IEEE Trans. Inf. Theory*, vol. 34, no. 2, pp. 181–193, 1988.
- G. Dueck, "Maximal Error Capacity Regions are Smaller Than Average Error Capacity Regions for Multi-User Channels," *Probl. Contr. Inform. Theory*, vol. 7, no. 1, pp. 11–19, 1978.

References

- E T. Ericson, "Exponential Error Bounds for Random Codes in the Arbitrarily Varying Channel," *IEEE Trans. Inf. Theory*, vol. IT-31, pp. 42–48, 1985.
- G J.A. Gubner, "On the Deterministic-Code Capacity of the Multiple-Access Arbitrarily Varying Channel," *IEEE Trans. Inf. Theory*, vol. 36, no. 2, pp. 262–275, 1990.
- J.-H. Jahn, "Coding of Arbitrarily Varying Multiuser Channels," *IEEE Trans. Inf. Theory*, vol. IT-27, no. 2, pp. 212–226, 1981.
- M. Wiese, H. Boche, I. Bjelaković, and V. Jungnickel, "The Compound Multiple Access Channel with Partially Cooperating Encoders", *IEEE Trans. Inf. Theory*, vol. 57, no. 5, pp. 3045–3066, 2011.
- M. Wiese, H. Boche, "The Arbitrarily Varying Multiple-Access Channel with Conferencing Encoders", submitted to *IEEE Trans. Inf. Theory*, available at http://arxiv.org/abs/1105.0319, 2011.
- F.M.J. Willems, "Informationtheoretical Results for the Discrete Memoryless Multiple Access Channel," Ph.D. dissertation, Katholieke Universiteit Leuven, Belgium, 1982.
- F.M.J. Willems, "The Discrete Memoryless Multiple Access Channel with Partially Cooperating Encodrs," *IEEE Trans. Inf. Theory*, vol. 29, no. 3, pp. 441–445, 1983.