Free CR distributions

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The structure of the lecture

1. CR geometry
2. Free CR distributions
3. $\mathfrak{su}(n, n)$ geometry
4. Fefferman construction
5. Tractor construction

Regular CR distributions

For each real submanifold $M \subset \mathbb{C}^n$, the complex tangent subspace

$$T^{\mathbb{C}}M = TM \cap \overline{TM}$$

defines the CR ditribution D.
For generic cases, D is of constant rank and bracket generating.
We call them regular CR structures.
The associated graded tangent bundle

$$\mathrm{Gr}\, TM = TM/D \oplus D$$

carries the structure of a Lie algebra, which is called the *symbol algebra*.

Codimension one case

There are only very few dimensions and codimensions, where the symbol algebras have to be constant on connected components of M.

One of them is the hypersurface case, going back to Poincare (1905) and Cartan.

The Cartan's solution to the equivalence problem in the lowest dimension of 3–dimensional surfaces in $\mathbb{C}^2$ lead to the general Cartan–Tanaka theory (and also the more famous Chern–Moser paper).

The hypersurface case is most important for function theory and one of the fruitful approaches to the invariants was suggested by Fefferman building the natural circle bundle over the hypersurface M equipped with a **conformal** structure.

Higher codimensions

Good examples of higher codimension regular CR distributions are the Shilov boundaries of homogeneous domains.
But there is no general nice solution to Cartan's equivalence problem here, because of the many nonisomorphic symbol algebras.
There are exceptions, however:

- 6–dimensional M in $\mathbb{C}^4$ (stable symbols are elliptic or hyperbolic)
- $(2n+n^2)$–dimensional M in $\mathbb{C}^{n+n^2}$
  i.e. $M_3 \subset \mathbb{C}^2$, $M_8 \subset \mathbb{C}^6$, $M_{15} \subset \mathbb{C}^{12}$, etc.

In both cases there are only very few isomorphism classes of the symbol algebra for dimensional reasons.

The 6–dimensional case was studied in detail in:

G. Schmalz, G; JS, The geometry of hyperbolic and elliptic CR–manifolds of codimension two, Asian Journal of Mathematics 4, Nr. 3 (2000), 565-598.

while the free CR case has been published this year:

G. Schmalz, G; JS, Free CR distributions, Central European Journal of Mathematics, Vol 10, 5 (2012), 1896-1913, DOI: 10.2478/s11533-012-0090-y

Cartan connections

Cartan geometries of type G/P are 'curved deformations' of the homogeneous space $G \to G/P$ with the Maurer–Cartan form $\omega \in \Omega^1(G;\mathfrak{g})$.

Definition

Cartan connection is an absolute parallelism $\omega \in \Omega^1(\mathcal{G},\mathfrak{g})$ on a principal fiber bundle $\mathcal{G} \to M$ with structure group P, enjoying nice invariance properties with respect to the principal action of P:

- $\omega(\zeta_X)(u) = X$ for all $X \in \mathfrak{p}$, $u \in \mathcal{G}$ (the connection reproduces the fundamental vertical fields)
- $(r^b)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$ (the connection form is equivariant with respect to the principal action)
- $\omega_{|T_u\mathcal{G}} : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$ (the absolute parallelism condition).

The free CR distributions

Consider the grading $\mathfrak{su}(n+1,n) = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$

$$
\begin{array}{cccc}
n & 1 & n & \\
\left(\begin{array}{c} \boxed{0} \\ \\ \boxed{-1} \\ \\ \boxed{-2} \end{array}\right. &
\begin{array}{c} \boxed{1} \\ \\ \boxed{0} \\ \\ \boxed{-1} \end{array} &
\left.\begin{array}{c} \boxed{2} \\ \\ \boxed{1} \\ \\ \boxed{0} \end{array}\right) &
\begin{array}{c} n \\ \\ 1 \\ \\ n \end{array}
\end{array}
$$

Clearly, the Lie bracket on the graded algebra $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ provides the unique possibility for the symbol up to isomorphisms and so the general theory implies, cf. the book
A. Cap; JS, Parabolic Geometries I, AMS, Math. Surv. Monogr. 154, 2009.

Indeed, the corresponding parabolic geometry is induced on generic surfaces of CR real codimension n^2 in $\mathbb{C}^{n+n^2}$:
The only nonzero component of the first cohomology $H^1(\mathfrak{g}_-, \mathfrak{su}(n+1,n))$ is of homogeneity -1, thus the entire geometry is determined by the filtration only. This is explained by

Lemma

The only complex structures $\tilde{J}$ on $\mathfrak{g}_{-1}$ which make the Lie bracket $\Lambda^2\mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ into a totally real antisymmetric form valued in skew Hermitian matrices are $\tilde{J} = \pm J$, where J is the standard complex structure on $\mathfrak{g}_{-1}$. Moreover, all linear homomorphisms $A \in GL(\mathfrak{g}_{-1})$ allowing an extension $\tilde{A}$ to a Lie algebra automorphism of $\mathfrak{g}_-$ are complex linear or complex anti–linear.

The entire second cohomology $H^2(\mathfrak{g}_-, \mathfrak{su}(n+1,n))$ lives in homogeneities zero and one (except cases $n = 1$ and $n = 2$). However, the zero homogeneity does not appear on the embedded submanifolds in $\mathbb{C}^{n+n^2}$. The remaining curvature is a torsion, similarly to free distributions.

Abstract definition for the 'integrable case'

Definition (Free CR distributions)

Consider a smooth manifold M of real dimension $2n+n^2$ equipped with a $2n$–dimensional distribution $D = T^{-1}M \subset TM$, such that $[D, D] = TM$. We call D a free CR distribution of dimension n on M if there is a fixed almost complex structure J on M such that the algebraic Lie bracket $\mathcal{L} : D \wedge D \to TM/D$ is totally real.

Lemma

Every free CR distribution of dimension n provides a regular infinitesimal flag structure of type $(PSU(n+1, n), P)$ on the $(2n+n^2)$–dimensional manifold M.

Thus, the Cartan–Tanaka– .. theory implies:

Theorem

For each free CR distribution of dimension n on a manifold M, there is the unique regular normal Cartan connection of type (G, P) on M (up to isomorphisms).
The only fundamental invariants of free CR distributions of dimensions $n > 2$ are concentrated in the curvature of homogeneity degree 1 and correspond to the totally trace-free part of the $\mathfrak{sl}(n, \mathbb{C})$-submodule $\mathrm{Hom}(\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-2}, \mathfrak{g}_{-2})$ in the torsion.
In the case $n = 2$, the same fundamental invariant exists and, aditionally, there is the Nijenhuis tensor of the complex structure J on the distribution, which vanishes automatically on the embedded real 8–dimensional manifolds M in $\mathbb{C}^6$.
Moreover, every smooth map between two free CR distributions respecting the distributions is either a CR morphism or a conjugate CR morphism.

The homogeneous model

Let us consider the Grassmannian of n-dimensional subspaces of $\mathbb{C}^{2n+1}$ and denote by Q its subset consisting of the isotropic subspaces.

Lemma

Q is a homogeneous CR-manifold of CR-dimension n and CR-codimension n^2 with transitive action of $SU(n+1,n)$. The kernel of the action is $\mathbb{Z}_{2n+1}$ and so the effective homogeneous model is $Q = G/P$, where $G = PSU(n+1,n) = SU(n+1,n)/\mathbb{Z}_{2n+1}$ and P is the isotropic subgroup of one fixed isotropic plane V_0 in Q.

Corollary

The group of all automorphisms of the homogeneous quadric Q is $PSU(n+1,n)$.

Exterior calculus – indication how to proceed ...

Let $\{X_i, X_{\bar{i}}, X_{i\bar{j}}\}$ be a 'suitable' local frame of $TM \otimes \mathbb{C}$, $\{\theta^i, \theta^{\bar{i}}, \theta^{j\bar{k}}\}$ the dual coframe, and $\{\theta^i, \theta^{[j\bar{k}]}\}$ be its restriction to TM. $D^\perp$ is generated by $\theta^{[j\bar{k}]}$.
Our choice yields structure euqations

$$
\begin{aligned}
d\theta^r =& f^r_{ij\bar{k}}\theta^i \wedge \theta^{[j\bar{k}]} + f^r_{\bar{i}j\bar{k}}\theta^{\bar{i}} \wedge \theta^{[j\bar{k}]} + f^r_{i\bar{j}k\bar{l}}\theta^{[i\bar{j}]} \wedge \theta^{[k\bar{l}]}, \\
d\theta^{[r\bar{s}]} =& \theta^r \wedge \theta^{\bar{s}} + f^{r\bar{s}}_{ij\bar{k}}\theta^i \wedge \theta^{[j\bar{k}]} + f^{r\bar{s}}_{\bar{i}j\bar{k}}\theta^{\bar{i}} \wedge \theta^{[j\bar{k}]} + f^{r\bar{s}}_{i\bar{j}k\bar{l}}\theta^{[i\bar{j}]} \wedge \theta^{[k\bar{l}]}
\end{aligned}
$$

where $f^r_{ij\bar{k}}$, $f^r_{\bar{i}j\bar{k}}$, $f^r_{i\bar{j}k\bar{l}}$, $f^{r\bar{s}}_{ij\bar{k}}$, $f^{r\bar{s}}_{\bar{i}j\bar{k}} = \overline{f^{s\bar{r}}_{ik\bar{j}}}$, $f^{r\bar{s}}_{i\bar{j}k\bar{l}}$ are the structure functions of the coframe $\{\theta^i, \theta^{\bar{i}}, \theta^{[j\bar{k}]}\}$ on M, which are uniquely determined by the choice of the complex frame X_i. Moreover, the functions $f^r_{ij\bar{k}}$ are symmetric in i, j.

Fixing the homogeneity zero

We know about the canonical Cartan connection $\omega\colon T\mathcal{G}\otimes\mathbb{C}\to\mathfrak{g}\otimes\mathbb{C}$ on $\pi\colon\mathcal{G}\to M$ and so any section $s:M\to\mathcal{G}$ provides the coframe:

$$s^*\omega=\begin{pmatrix}\omega^i_j & \omega_i & \omega_{[i\bar{j}]}\\ \omega^j & -2i\,\mathrm{Im}\,\mathrm{tr}\,\omega^i_j & -\omega_{\bar{i}}\\ \omega^{[i\bar{j}]} & -\omega^{\bar{j}} & -\omega^{\bar{j}}_{\bar{i}}\end{pmatrix}$$

and our task is to improve our choices in such a way to meet the curvature properties of ω.
Fixing the G_0 freedom of s, we may assume

$$\omega^i=\theta^i \mod D^\perp$$
$$\omega^{[i\bar{j}]}=\theta^{[i\bar{j}]}.$$

1st prolongation

Let us write the general ansatz:

$$\omega^i = \theta^i + C^i_{j\bar{k}}\omega^{[j\bar{k}]},$$
$$\omega^i_j = A^i_{kj}\omega^k + B^i_{\bar{k}j}\omega^{\bar{k}} \mod D^\perp$$

for some functions A^i_{kj}, $B^i_{\bar{k}j}$, $C^i_{j\bar{k}}$. Notice that the C's provide the splitting of $TM = D \oplus Q$, while the A's and $B's$ define a partial connection (similar to homogeneity one part of Webster–Tanaka connection).

The $\exp \mathfrak{g}_1$ freedom for s allows just for killing the A^i_{ik} trace.

Curvatures

$$\Omega^{[i\bar{j}]} \equiv P^{i\bar{j}}_{rs\bar{t}}\omega^r \wedge \omega^{[s\bar{t}]} + \overline{P^{j\bar{i}}_{rt\bar{s}}}\omega^{\bar{r}} \wedge \omega^{[s\bar{t}]} \quad \text{mod } \wedge^2 D^\perp,$$

where

$$P^{i\bar{j}}_{rs\bar{t}} = f^{i\bar{j}}_{rs\bar{t}} + A^i_{rs}\delta^{\bar{j}}_{\bar{t}} + \overline{B^j_{\bar{r}t}}\delta^i_s + \overline{C^j_{t\bar{s}}}\delta^i_r.$$

$$\Omega^i = Q^i_{rs}\omega^r \wedge \omega^s + Q^i_{r\bar{s}}\omega^r \wedge \omega^{\bar{s}} \quad \text{mod } D^\perp$$

where

$$Q^i_{[rs]} = A^i_{[rs]} - \frac{1}{2}(A^j_{rj}\delta^i_s - A^j_{sj}\delta^i_r) + \frac{1}{2}(\overline{B^j_{\bar{r}j}}\delta^i_s - \overline{B^j_{\bar{s}j}}\delta^i_r)$$

$$Q^i_{r\bar{s}} = C^i_{r\bar{s}} - B^i_{\bar{s}r} - \overline{A^j_{sj}}\delta^i_r + B^j_{\bar{s}j}\delta^i_r.$$

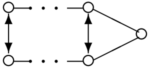
Conclusion in homogeneity one

Vanishing of Q and vanishing of all traces in P provide exactly enough information to compute all $A's$, $B's$, and $C's$ explicitly from the structure equations now.

What next? Homogeneity two considerations would exploit the freedom in $\exp \mathfrak{g}_2$ and the explicite knowledge of the curvature of the Cartan connection to complete the remaining Christoffel symbols of the chosen linear connection on M and the first part of the so called P–tensor. And so on ...

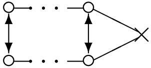
An alternative version is to mimic the Fefferman construction of the **conformal** structure on the canonical circle bundle over the hypersurface type CR geometries. This provides a much better understood $|1|$–graded geometry and much information on the original CR structure.

The Satake diagram for the algebra $\mathfrak{g} = \mathfrak{su}(n, n)$ is



and the possible parabolic subalgebras are given by placing properly croses over the nodes (the black ones must not be touched and the arrows bring further limitations).

At the same time, the length of the grading is given by summing the coefficients corresponding to the crossed nodes in the expression for the highest weight of the adjoint representation in terms of the simple roots. Clearly, the only possible $|1|$–grading on $\mathfrak{g}$ is given by



The chosen grading of $\mathfrak{su}(n,n) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is as follows

$$\begin{array}{cc} n & n \end{array}$$
$$\begin{pmatrix} \boxed{0} & \boxed{1} \\ \boxed{-1} & \boxed{0} \end{pmatrix} \begin{array}{c} n \\ n \end{array}$$

If we choose the Hermitian from in the split signature properly, we immediately see that $\mathfrak{g}_{\pm 1}$ are the spaces of skew-Hermitian matrices with respect to the anti-diagonal (dual to each other), while $\mathfrak{g}_0 = \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}$ (where always couple of matrices $(A, -\bar{A}^T)$ are in the block diagonal, with $\operatorname{Tr} A \in \mathbb{R}$).

For all $|1|$–graded parabolic geometries, the entire structure is given by the appropriate reduction of the frame bundle to the structure group G_0.
We shall work with $G = SU(n, n)$ and so G_0 will be the group of complex matrices with real determinant.
Thus the structure group G is a double covering of the effective group $G/\mathbb{Z}_2$ coming from the effective Klein geometry of this type.

Clearly, the effective geometry is given by identifying the tangent bundle TM to a real n^2–dimensional manifold M with the space

$$\Lambda_{\text{skew-H}}(\mathcal{S})$$

of an auxiliary complex n–dimensional bundle $\mathcal{S}$ over M. (Notice, that only real action of the centre in G_0 occurs on the skew-Hermitian matrices.)
We shall write $\mathbb{S} \simeq \mathbb{C}^n$ for its standard fiber.

There are two general functorial constructions on Cartan geometries $(\mathcal{G}, \omega)$

- the correspondence spaces
- the stucture group extensions

The first one is given by a choice of subgroups $Q \subset P \subset G$ and it always increases the underlying manifold $M = \mathcal{G}/P$ into a fiber bundle $\tilde{M} = \mathcal{G}/Q \to M$ with fiber Q/P.
The other one is based on embeddings of the structure group $G \to \tilde{G}$ and reasonable choices of subgroups $P \subset G$, $\tilde{P} \subset \tilde{G}$, and it leads to Cartan geometries on the same manifolds M, but with bigger structure groups.
Combination of these two steps yields the Fefferman–like constructions.

Constructions for homogeneous models

Let G/P and $\tilde{G}/\tilde{P}$ be two (real or complex) parabolic homogeneous spaces. Assumptions:

- fixed homomorphism $i\colon G \to \tilde{G}$ which is infinitesimally injective
- the G–orbit of $o = e\tilde{P} \in \tilde{G}/\tilde{P}$ is open (thus, $\mathfrak{g} \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ induced by $i' : \mathfrak{g} \to \tilde{\mathfrak{g}}$ is surjective).
- $P \subset G$ contains $Q := i^{-1}(\tilde{P})$.

consequently:

- there is the natural projection $\pi : G/Q \to G/P$
- Q is a closed subgroup of G (which is usually not parabolic)
- the homomorphism $i : G \to \tilde{G}$ induces the smooth map $G/Q \to \tilde{G}/\tilde{P}$ which is a covering of the G–orbit of o,
- the latter open subset in $\tilde{G}/\tilde{P}$ carries a canonical geometry of type $(\tilde{G}, \tilde{P})$. This can be pulled back to obtain such a geometry on G/Q.

If we use the quite obvious emebdding of $SU(p+1, q+1)$ into the orthogonal group, we get the original Fefferman construction for the hypersurface type CR in the homogeneous case.
As well known, there is just the imaginary part of the center of the Levi component $\mathfrak{g}_0$ in $\mathfrak{p}$ which is not in $\mathfrak{q}$.
In order to obtain $\tilde{M}$, let us first factor out the action of all the P, except the complex one-dimensional centre of G_0. We get a complex one-dimensional bundle over M and we immediately see that this bundle can be understood as the complex line bundle corresponding to the action $z \cdot s = zs$ by the central element $z \in \mathbb{C}$. This bundle is usually denoted by $\mathcal{E}(1, 0)$.
Now $\tilde{M}$ is clearly the quotient of this bundle by the action of the real part of $\mathbb{C}$, thus the bundle of lines in $\mathcal{E}(1, 0)$. Thus G/Q turns out to be a circle bundle over G/P.
The functorial character of the above construction ensures that it survives in the curved setting without any modifications (except we need the existence of the bundle $\mathcal{E}(1, 0)$).

Inclusions of parabolic geometries

Especially, it may happen that $i(G)\tilde{P} = \tilde{G}$ and $i(P) = i(G) \cap \tilde{P}$ i.e. $Q = P$ is the parabolic subgroup.

Then both parabolic geometries turn out to live over the same base manifold $G/P = \tilde{G}/\tilde{P}$. We say that i is an *inclusion of parabolic homogeneous spaces*.

In fact the spinorial geometry mentioned above provides one of these very rare examples, appearing as Fefferman spaces for the so called free rank ℓ distributions. The lowest dimensional case of generic 3-dimensional distributions on 6-dimensional manifolds M leads to conformal structures with split signature (shown by Robert Bryant in another way very long ago), the higher dimensions were settled recently [B. Doubrov, J.S. Inclusions of parabolic geometries, Pure and Applied Mathematics Quaterly 6, 3 (2010), Special Issue: In honor of Joseph J. Kohn, Part 1, 755–780.].

In our case, the free CR–distributions come from the grading $\mathfrak{su}(n+1,n) = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ of the form

$$
\begin{array}{cccc}
n & 1 & n & \\
\left(\begin{array}{ccc} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{array}\right) & & & \begin{array}{c} n \\ 1 \\ n \end{array}
\end{array}
$$

Next, we obtain an embedding to a $|1|$–graded Lie algebra:

$$\begin{pmatrix} A & X & Y \\ -Z^* & 2\alpha & -X^* \\ T & Z & -A^* \end{pmatrix} \mapsto \begin{pmatrix} A & \frac{1}{\sqrt{2}}X & \frac{1}{\sqrt{2}}X & Y \\ -\frac{1}{\sqrt{2}}Z^* & \alpha & \alpha & -\frac{1}{\sqrt{2}}X^* \\ -\frac{1}{\sqrt{2}}Z^* & \alpha & \alpha & -\frac{1}{\sqrt{2}}X^* \\ T & \frac{1}{\sqrt{2}}Z & \frac{1}{\sqrt{2}}Z & -A^* \end{pmatrix}$$

where $A, Y, T \in \mathrm{Mat}_\ell(\mathbb{C})$, $X, Z \in \mathbb{C}^\ell$, $Y + Y^* = T + T^* = 0$, $\alpha = -i\,\mathrm{Im}\,\mathrm{Tr}\,A$.

We shall consider the subalgebra $\tilde{\mathfrak{p}}$ corresponding to the only $|1|$–graded geometry.

The homogeneous case

Similarly to the hypersurface case, the preimage $\mathfrak{q}$ of $\tilde{\mathfrak{p}}$ is nearly the entire $\mathfrak{p}$, with just one dimension in the centre $\mathbb{C}$ of $\mathfrak{g}_0$ lacking.

We again can first consider the quotient of G by all of the P but the centre of G_0. This will provide a complex line bundle $\mathcal{E}(1,0)$ and, again, the requested space G/Q can be identified with the bundle of lines in $\mathcal{E}(1,0)$. Thus we have obtained a circle bundle again.
This fully survives for the curved free CR–distributions.

The key to an explicit construction is the standard tractor calculus. The *standard tractor bundle* is $\mathcal{T}M = \mathcal{G} \times_P \mathbb{V}$, where $\mathbb{V}$ is the standard $SU(n+1,n)$ representation. The filtration $\mathbb{V} = \mathbb{V}^0 \supset \mathbb{V}^1 \supset \mathbb{V}^2 \supset 0$ induces the filtration of the tractor bundle $\mathcal{T}M = \mathcal{T}^2M \supset \mathcal{T}^1M \supset \mathcal{T}^0M \supset 0$ on $\mathcal{T}M$.

As a G_0 module, the standard representation splits as $\mathbb{V} = \mathbb{C}^n \oplus \mathbb{C} \oplus \mathbb{C}^n$ and the G_0–module $\mathbb{S} = \mathbb{V}/\mathbb{V}^2 = \mathbb{C} \oplus \mathbb{C}^n$ has the property that $\mathfrak{g}_-$ equals to the space of skew–Hermitian 2-forms on $\mathbb{S}$. It is easy to see, that this G_0 module structure is compatible with the inclusion $G_0 \to \tilde{G}_0$.

In order to recover this construction for a general Cartan geometry of the given type, we have to know the splitting of $\mathcal{S}$, but this is obtained after the first prolongation already.

On any free CR–distribution, the definition of a covering by the $SU(n+1, n)$ geometry is necessary for the existence of the standard tractor bundle.
Clearly, this is equivalent to the existence of the complex line bundle $\mathcal{E}(1, 0)$.
But for embedded free CR–manifolds this is simple to construct similarly to the hypersurface case: There is the trivial canonical subbundle $\mathcal{K}$ there, defined as the $(n+1)$–exterior power of the anihilator of the holomorphic vectors in the complexified tangent bundle. Clearly the centre in G_0 acts by the power z^{-n-2} and so we can take the appropriate root and consider the dual space.
Here we shall have the canonical $(n+n^2)$–exterior power of the analogous anihilator. This conludes the construction of the geometric structure on the Fefferman space.

The normality question

In general, the canonical normal Cartan connection on the Fefferman space does not need to be the one induced from the functorial construction.
In our case, the curvature κ allows an invariant projection $\kappa_{1,1}$ defined by restriction to arguments in $\mathfrak{g}_{-1}$ only.

Theorem

The Fefferman extension of a free CR–geometry to $|1|$–graded geometry is normal if and only if $\kappa_{1,1}$ vanishes identically.

Proof is based on the fact that κ is coclosed and is concentrated in the positive homogeneity components in the space $\mathrm{Hom}(\wedge^2(\mathfrak{g}_-), \mathfrak{g})$. (Quite tedious computation.)

By the general BGG machinary, we know, that the entire curvature is given by a natural linear operator applied to the harmonic part of the curvature. Since our projection to the component $\kappa_{1,1}$ is invariant two, every projection of $\kappa_{1,1}$ to an irreducible component would be an invariant operator, too. But such operators cannot exist by linear operators without curvatur in its symbol by Kostant.

The Bianchi identity relates the differential and the fundamental derivative

$$\partial\kappa = \sum_{\text{cycl}} i_\kappa \kappa - D\kappa$$

and employing $\partial^*\kappa = 0$ we obtain for the lowest homogeneity component

$$\Box\kappa^0_{1,1} = \partial^* \sum_{\text{cycl}} i_\kappa \kappa.$$

(Example similar to Stuart Armstrong, arXiv:0708.3027v3)
The $2n + n^2$-dimensional flat free CR manifold Q can be described in coordinates $\{z_j, w_{kl}\}$ with $1 \leq j \leq n$, $1 \leq k \leq l \leq n$, where $z_j, w_{kl} \in \mathbb{C}$ and $\Re w_{kk} = 0$ for $1 \leq k \leq n$ by the $D^{(1,0)}$ vector fields

$$Z_j = \frac{\partial}{\partial z_j} - \sum_{p=j}^{n} \bar{z}_p \frac{\partial}{\partial w_{jp}}.$$

Then

$$W_{kk} = [Z_k, \bar{Z}_k] = \frac{\partial}{\partial w_{kk}} - \frac{\partial}{\partial \bar{w}_{kk}}$$

$$W_{kl} = [Z_k, \bar{Z}_l] = \frac{\partial}{\partial w_{kl}} \qquad \text{if } k < l$$

$$W_{kl} = [Z_k, \bar{Z}_l] = -\frac{\partial}{\partial \bar{w}_{lk}} \qquad \text{if } k > l.$$

For $n \geq 4$ we modify Q by replacing Z_1 by

$$Z_1' = Z_1 + \bar{w}_{12}\frac{\partial}{\partial w_{34}}.$$

Notice that $[Z_1', Z_j] = 0$ for $2 \leq j \leq n$, hence the modified CR structure is still integrable and $[Z_1', \bar{Z}_j] = W_{1j}$. The only resulting change in the structure equations is that now $f_{1[12]}^{[34]} = 1$. It follows that the tensor P is already trace-free for $A = B = 0$, hence $A = B = C = 0$ in this case and the only non-vanishing coefficient in P is $P_{1[34]}^{[12]} = 1$. Since the curvature in homogeneity 1 is constant, by the Bianchi identity, the curvature of higher homogeneity vanishes automatically.