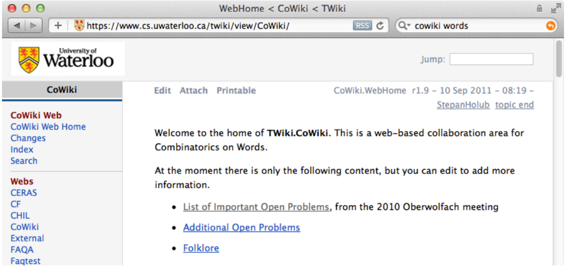
Avoidability under Permutations

Florin Manea Mike Müller Dirk Nowotka

Department of Computer Science
Christian-Albrechts-Universität zu Kiel
Germany

BIRS 2012

Combinatorics on Words Wiki



https://www.cs.uwaterloo.ca/twiki/view/CoWiki/

or

google **cowiki words**

or

go to link on **Jeff Shallit's home page**

Avoidability under Permutations

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Avoidability

Pattern p: word over $\{x, y, \ldots\}$

$$xxy$$

$\mathbf{w}$ avoids p if $\sigma(p)$ does not occur in $\mathbf{w}$ for all non-erasing morphisms σ

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Generalization: functional dependencies between variables

Pattern p: word over $\{x, y, \ldots, f(x), g(x), f(y), \ldots\}$

$$x\,f(x)\,y$$

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We consider **permutations** here.

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$$x\, f(x)\, y$$

We consider **permutations** here.

$\mathbf{w}$ avoids p if u does not occur in $\mathbf{w}$, where u results from p after

- all variables x are replaced by $\sigma(x)$ and
- all $f(x)$ are replaced by $f'(\sigma(x))$
- for all non-erasing morphisms σ and permutations f' on the alphabet.

An Example

$$x f(x) x$$

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$$xf(x)x$$

is avoidable in Σ_3 (but not in Σ_2).

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Consider

$$\mathbf{v} = \delta(\mathbf{t}) = 02110022100221002110\ldots$$

where

$$\delta(0) = 02110$$
$$\delta(1) = 02210$$

and $\mathbf{t}$ is the Thue-Morse word.

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Lemma $(*)$

$\mathbf{v}$ *avoids the pattern* $xf(x)x$.

Another Example

$$xf^5(x)f^{12}(x)$$

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- unavoidable in Σ_8

Another Example

$$xf^5(x)f^{12}(x)$$

- unavoidable in Σ_2
- avoidable in Σ_4 (witness on next slide)
- unavoidable in Σ_8
- …in fact, avoidable in Σ_m iff $m \in \{3, \ldots 7\}$

Another Interesting Word

Consider

$$\mathbf{u} = \delta(\mathbf{t}) = 012013213012031023012013213\ldots$$

where

$$\delta(0) = 012013213$$
$$\delta(1) = 012031023$$

and $\mathbf{t}$ is the Thue-Morse word.

Claim $\mathbf{u}$ avoids $xf^5(x)f^{12}(x)$

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Lemma $(**)$

- $\mathbf{u}$ *contains no* $vf(v)g(v)$ *for all* $|v| \geq 7$
- $\mathbf{u}$ *contains no* $wf^i(w)f^j(w)$ *with*

$$|\{w_{[\ell]}, f^i(w)_{[\ell]}, f^j(w)_{[\ell]}\}| \leq 2$$

for all $\ell \leq |w| \leq 6$*.*

Result

Theorem

Let $p = xf^i(x)f^j(x)$ with $i \neq j$. We can effectively determine the values m such that p is avoidable over Σ_m.

Let ...

$$\begin{aligned} k_1 &= \min\{t \text{ with } t \nmid |i-j| \text{ and } t \nmid i \text{ and } t \nmid j\} \\ k_2 &= \min\{t \text{ with } t \mid |i-j| \text{ and } t \nmid i \text{ and } t \nmid j\} \\ k_3 &= \min\{t \text{ with } t \mid i \text{ and } t \nmid j\} \\ k_4 &= \min\{t \text{ with } t \nmid i \text{ and } t \mid j\} \end{aligned}$$

(minimum of empty set equals $+\infty$ here)

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Example

pattern $xf^5(x)f^{12}(x)$

$$k = k_1 = 8, \qquad k_2 = 7, \qquad k_3 = 5, \qquad k_4 = 2$$

Cases $4 \leq m$

Lemma

The pattern $xf^i(x)f^j(x)$, with $i \neq j$, is

1. *avoidable over Σ_m if $4 \leq m < k$ and*
2. *unavoidable over Σ_m if $k \leq m$.*

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pattern $xf^5(x)f^{12}(x)$

- avoidable over Σ_m if $4 \leq m < 8$ and
- unavoidable over Σ_m if $8 \leq m$.

Case $4 \leq m < k$

$$k_1 = \min\{t \mid t \nmid |i-j|, t \nmid i, t \nmid j\}, \quad k_3 = \min\{t \mid t \mid i, t \nmid j\},$$
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Case study on $\min\{k_1, k_2, k_3, k_4\}$.

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Case study on $\min\{k_1, k_2, k_3, k_4\}$.

For example, let $k_4 = \min\{k_1, k_2, k_3, k_4\}$.
$4 \leq m < k = k_1$ and $k_4 \leq k$ implies

- $\mathrm{ord}_f(a)|i$ or $\mathrm{ord}_f(a)|j$ and
- for every factor $uf^i(u)f^j(u)$ and every position ℓ in u we have $u_{[\ell]} = f^i(u)_{[\ell]}$ or $u_{[\ell]} = f^j(u)_{[\ell]}$

Avoidable by Lemma $(**)$.

Case $k \leq m$

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Suppose $k = \max\{k_1, k_2\} \leq m$.

A word avoiding p must avoid cubes and abc and abb (a, b, c different).

… and the cases $m = 2$ and $m = 3$ …

		j(mod 6)						
		0	1	2	3	4	5	
i(mod 6)	0	Y	N	Y	N	Y	N	2 letters
		Y	Y	Y	Y	Y	Y	3 letters
	1	N	N	N	N	N	N	2 letters
		Y	Y	N	N	N	Y	3 letters
	2	Y	N	Y	N	Y	N	2 letters
		Y	N	Y	N	Y	Y	3 letters
	3	N	N	N	N	N	N	2 letters
		Y	Y	N	Y	N	Y	3 letters
	4	Y	N	Y	N	Y	N	2 letters
		Y	Y	Y	N	Y	N	3 letters
	5	N	N	N	N	N	N	2 letters
		Y	N	N	N	N	Y	3 letters

2 letters: avoidance iff $i \equiv j \equiv 0 \pmod 2$

3 letters: avoidance by some cube-free ternary word or word $\mathbf{v}$ from Lemma $(*)$.

Actually

Theorem

Let $p = f^i(x)f^j(x)f^k(x)$*. We can effectively determine the values* m *such that* p *is avoidable over* Σ_m*.*

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Remark

All results hold for both morphic and antimorphic extensions of the permutations.

Example for Antimorphic Case

Consider

$$\mathbf{w} = \delta(\mathbf{t}) = 001102211001221100122001102 2\ldots$$

where

$$\delta(0) = 0011022$$
$$\delta(1) = 1100122$$

and $\mathbf{t}$ is the Thue-Morse word.

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— End of Talk —

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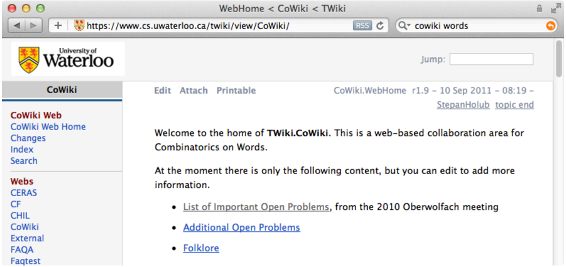

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