Kueker's conjecture

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Kueker's conjecture is true for

- (Buechler) T superstable.
- (Hrushovski) T stable.
- (Hrushovski) T interpreting a linear order.
- (Hrushovski) T with built-in Skolem functions.

T is a **Kueker theory** if every uncountable model of T is $\aleph_0$-saturated and T is not $\aleph_0$-categorical.

Restated Kueker's Conjecture

Every Kueker theory is $\aleph_1$-categorical.

Theorem

If T is a Kueker theory and $\mathrm{dcl}(\emptyset)$ is infinite then T does not have the strict order property.

Combining with the stable case we obtain:

Corollary

Kueker's conjecture is true for NIP theories with infinite $\mathrm{dcl}(\emptyset)$.

Plan of the proof of the conjecture:

Assuming that T is a Kueker theory:

(A) Find a strongly minimal formula $\phi(x)$.

(B) Prove that T has neither SOP nor IP .

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Assuming that $|\mathrm{dcl}(\emptyset)| = \aleph_0$ we prove:

Theorem

(A)' Strongly minimal types are dense (every non-algebraic formula is contained in a strongly minimal type).

(B)' T is NSOP.

Hrushovski

Let T be a Kueker theory.

- T is small ($|S(\emptyset)| \leq \aleph_0$).
- T cannot have an uncountable model which is atomic over a finite subset
- The prime model over a finite set is minimal (If the prime model was not minimal then we could find an uncountable atomic model.)

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- T is small ($|S(\emptyset)| \leq \aleph_0$).
- T cannot have an uncountable model which is atomic over a finite subset
- The prime model over a finite set is minimal (If the prime model was not minimal then we could find an uncountable atomic model.)
- Almost minimal formulas are dense in any model prime over a finite set.

Definition

- $\phi(x, \bar{a})$ **is almost minimal over** $A \supseteq \bar{a}$ if there are infinitely many algebraic types and a unique non-algebraic complete type over A containing it;
- A **complete type is almost minimal** if it is non-algebraic and contains an almost minimal formula.

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Theorem (A)'

If T is a Kueker theory and if $\mathrm{dcl}(\emptyset)$ is infinite then any almost minimal formula has Morley rang 1.

- A first-order structure is **almost minimal** if it has infinitely many algebraic and a unique non-algebraic 1-type; in particular, $\mathrm{acl}(\emptyset) = M$.
- Equivalently, $x = x$ is almost minimal over $\emptyset$.
- There are two types of almost minimal structures and according to the multiplicity of the unique non-algebraic 1-type $p \in S_1(\emptyset)$:

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- There are two types of almost minimal structures and according to the multiplicity of the unique non-algebraic 1-type $p \in S_1(\emptyset)$:

Case 1. $\mathrm{mult}(p) < \aleph_0$

- If $\mathrm{mult}(p) = 1$ then M is minimal (any definable with parameters subset is either finite or co-finite).
- M is the union of $\mathrm{mult}(p)$-many (domains of) minimal structures (after naming a bit of $\mathrm{acl}(\emptyset)$).

Case 2. $\mathrm{mult}(p) = 2^{\aleph_0}$

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Confirmation of the following would complete part (A) of the Plan.

Conjecture 1. This case does not happen in Kueker theories.

Conjecture 2.

(T any small theory) Suppose that $(\mathcal{C},\ldots)$ is an almost minimal structure of the second type. Then exactly one of the following two options holds:

(I) Every non-algebraic $p \in S_1(\mathcal{C})$ is definable and its unique global heir is generically stable.

(II) There is a proper definable (with parameters) partial order on elements of $\mathcal{M}$.

Assume that T is a Kueker theory and that $\mathcal{C} = \mathrm{dcl}(\emptyset)$ is infinite. We prove that any almost minimal formula has Morley rank 1.

Proof ingredients

1. Prove that every type over a finite domain has finite multiplicity, so every almost minimal formula is a finite union of minimals.

2. Apply the Dichotomy theorem for minimal structures (formulas).

3. Eliminate the asymmetric case.

4. Using 'regularity' properties of symmetric minimal structures derive strong minimality.

- By a $\mathcal{C}$-**type** over A we mean a complete, non-algebraic type over A which is finitely satisfiable in $\mathcal{C}$ (every formula from the type is satisfied by a tuple of elements of $\mathcal{C}$).

- $\{\bar{a}_i \mid i < \alpha\}$ is a $\mathcal{C}$-**sequence over** A if $\mathrm{tp}(\bar{a}_i / A \cup \{\bar{a}_j \mid j < i\})$ is a $\mathcal{C}$-type for all $i < \alpha$.

- B is **almost atomic over** A if for all $\bar{b} \in B$ there is a finite $A_0 \subset A$ such that $\mathrm{tp}(\bar{b}/A')$ is isolated for all finite $A_0 \subset A' \subset A$.

- In any small theory almost atomic models over countable sets exist.

If $p(x) = \mathrm{tp}(\bar{a}/A) \in S(A)$ is a $\mathcal{C}$-type and $q = \mathrm{tp}(\bar{b}/A) \in S(A)$ is isolated then

$$p \overset{w}{\perp} q \quad \text{(i.e. } p(\bar{x}) \cup q(\bar{y}) \text{ determines a complete type).}$$

Lemma

Suppose that $I = \{a_i \mid i < \alpha\}$ is a $\mathcal{C}$-sequence.

(a) If $\alpha \leq \omega_1$ then there is $M \supset I$ which is almost atomic over I.

(b) If $\alpha = \omega$ then there is a prime model M over I, and if I is indiscernible then M is $\aleph_0$-saturated.

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- Every isolated type has finite multiplicity; otherwise, there is an uncountable model atomic over a finite set.
- Every type over a finite domain has finite multiplicity.

Semi-isolation

In this section we do not assume that T is a Kueker theory!

- Let p be a type. For $a \in p(\mathcal{M})$ and $A \subseteq p(\mathcal{M})$ define $a \in \mathrm{Sem}_p(A)$, or a **is semi-isolated over** A, iff:

  there is $\phi(x) \in \mathrm{tp}(a/A)$ such that $\phi(x) \vdash p(x)$.

- This defines Sem_p as an operation on the power set of $p(\mathcal{M})$.

Dichotomy theorem for minimal structures

Let $(\mathcal{C}, \ldots)$ be a minimal first-order structure, let $\mathcal{M}$ be its saturated elementary extension and let $p \in S_1(\mathcal{C})$ be the unique non-algebraic 1-type. Then exactly one of the following two options holds:

(I) Every $\mathcal{C}$-sequence over $\mathcal{C}$ is symmetric (totally indiscernible). In this case $(p(\mathcal{M}), \mathrm{Sem}_p)$ is a pregeometry; p is definable, its unique global heir $\hat{p}$ is generically stable and $(\hat{p}, x = x)$ is strongly regular.

(II) There exists a $\mathcal{C}$-sequence which is not symmetric. In this case there is an infinite $\mathcal{C}_0 \subseteq \mathcal{C}$ directing a type (defined later) over some finite $E \subset \mathcal{M}$.

- We say that $\mathcal{C} \subset \mathrm{dcl}(A)$ **directs a type over** A if there is an A-definable partial order $\leq$ such that:

  (D1) $\{x \in \mathcal{C} \mid c \leq x\}$ is a co-finite subset of $\mathcal{C}$ for all $c \in \mathcal{C}$;

  (D2) $\mathcal{C}$ is an initial part or $\mathcal{M}$: $c \in \mathcal{C}$ and $a \leq c$ imply $a \in \mathcal{C}$.

- In this case we say that the partial type

$$p(x) = \{\phi(x) \mid \phi(x) \text{ is over } A \text{ and } \phi(\mathcal{C}) \text{ is co-finite in } \mathcal{C}\}.$$

  is $\mathcal{C}$**-directed over** A, or $(\mathcal{C}, \leq)$-directed over A.

- A type is **directed by constants** if it is $(\mathcal{C}, \leq)$-directed over A for some $\leq$ and some $\mathcal{C} \subset \mathrm{dcl}(A)$.

Suppose that $\mathcal{C}$ directs a type. $B \subset M$ is $\mathcal{C}$-**independent** if any finite subset can be arranged into a $\mathcal{C}$-sequence.

Theorem

Suppose that T is small. Then there is $\mathcal{C}_0 \subset \mathcal{C}$ and a finite A such that:

$B \subset M$ is $\mathcal{C}_0$-independent over A iff it is pairwise $\mathcal{C}_0$-independent over A.

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Theorem

Suppose that T is small. Then there is $\mathcal{C}_0 \subset \mathcal{C}$ and a finite A such that:

$B \subset M$ is $\mathcal{C}_0$-independent over A iff it is pairwise $\mathcal{C}_0$-independent over A.

- If $I = \{a_i \mid i < \omega_1\}$ is a $\mathcal{C}_0$-sequence over A and $M \supset AI \setminus \{a_0\}$ is almost atomic over $AI \setminus \{a_0\}$ then $tp(a_0/a_1)$ is not realized in M.
- There are no types directed by constants in a Kueker theory.
- Any almost minimal formula in a Kueker theory is a finite union of minimal formulas of symmetric type.

Assume: T is a Kueker theory, $\mathrm{acl}(\emptyset)$ is absorbed into the language, $\phi(x)$ is minimal over $\emptyset$ and $\mathcal{C} = \mathrm{dcl}(\emptyset \cap \phi(\mathcal{M}))$.

- $(C, \ldots)$ with the induced structure is symmetric: every $\mathcal{C}$-sequence (over $\emptyset$) is totally indiscernible and $(p(\mathcal{M}), \mathrm{Sem}_p)$ is a pregeometry.
- Let $I \subset \phi(\mathcal{M})$ be an indiscernible $\mathcal{C}$-sequence of size $\aleph_1$ and let $M \supset I$ be almost atomic over I.
- It suffices to prove that $\phi(\mathcal{M})$ cannot be definably split into two infinite subsets.

Suppose that $p \in S(\emptyset)$ is strongly minimal, non-isolated and that I is a countably infinite Morley sequence in p.

Corollary

T_I is small and the prime model is an $\aleph_0$-saturated model of T.

The positive answer to the following would prove (B).

Question

Does any unstable non-algebraic formula isolating a type over $\emptyset$ have such an extension over any finite domain?

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If T is a Kueker theory and $\leq$ is 0-definable (on elements).

(1) There is an isolated type whose locus is not an antichain.

(2) A type from (1) has such an extension over any finite super-domain.

(3) T has uncountable model atomic over a finite set.