Neostability Theory Meeting

From Finite to First-order Model Theory

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- Classes of finite structures
- "Intricacies" of the $\mathbb{K} \leftrightarrow \mathfrak{M}$ correspondence
- Examining $\mathbb{K}_\forall \to \mathbb{K}$

Some familiar classes

1. $\mathfrak{M}$ is an ultrahomogenous structure. $\mathbb{K}$ is the set of finite induced substructures of $\mathfrak{M}$ (its age).
2. $\mathfrak{M}$ is a smoothly approximable structure. $\mathbb{K}$ is the set of homogeneous substructures of $\mathfrak{M}$:

$$\mathcal{A} \leq_{hom} \mathcal{M} \Leftrightarrow Aut(\mathcal{A}) \text{ and } Aut(\mathfrak{M}/\{A\}) \text{ agree on } A^r,\ r < \omega$$

- In 1, many/most members of $\mathbb{K}$ are not much like $\mathfrak{M}$.
- In 2, members of $\mathbb{K}$ are just like $\mathfrak{M}$ only finite.

Somewhere in between

- L^k = formulas with at most k variables, free or bound.
- $\mathfrak{M}$ an $\aleph_0$-categorical structure with the finite sub-model property.
  $\mathbb{K}$ the set of finite L^k-elementary substructures of $\mathfrak{M}$.
- I just assume that:
  - $\mathbb{K}$ has JEP and AP/models.
  - Members of $\mathbb{K}$ are algebraically closed.
- In this case, $\mathfrak{M}$ is the direct limit of $\mathbb{K}$, but needn't be smoothly approximable. (L^k-types don't correspond to orbits.)

Normally, we'd just use $\mathcal{M}$ ...

- It's true that the model theory of $\mathbb{K}$ and that of $\mathfrak{M}$ are essentially identical.
- But, the model theory of $\mathbb{K}$ can at least be *expressed* independently

  ... and we can link that to properties of $\mathbb{K}$ that model theorists don't usually consider.
- $\mathbb{K}_\forall$ = all induced substructures of $\mathfrak{M}$. Each $A \in \mathbb{K}_\forall$ extends to some $\mathcal{B} \in \mathbb{K}$.

  How complex is this transformation $\mathbb{K}_\forall \to \mathbb{K}$?

þ-Rank in $\mathbb{K}$

- $\delta(\overline{y}, \overline{z})$, $\varphi(\overline{x}, \overline{y})$ boolean combinations of k-variable formulas, $1 < r < \omega$.
  $\pi(\overline{x})$ a type over $A \subseteq M_0$ for some $\mathcal{M}_0 \in \mathbb{K}$.
- $\text{þ}(\pi(\overline{x}), \varphi, \delta, r) \geq e + 1$ if there $\mathcal{M} \in \mathbb{K}_A$ and $\overline{c} \in M^{\overline{z}}$ such that:
  1. For every $\mathcal{N} \in \mathbb{K}_{A\overline{c}}$, $\left\{\varphi(\mathcal{N}, \overline{b}) : \mathcal{N} \vDash \delta(\overline{b}, \overline{c})\right\}$ is r-inconsistent.
  2. For every $n < \omega$, there is an $\mathcal{N} \in \mathbb{K}_{A\overline{c}}$ such that

$$| \left\{\varphi(\mathcal{N}, \overline{b}) : \text{þ}(\pi(\overline{x}) \wedge \varphi(\overline{x}, \overline{b}), \varphi, \delta, r) \geq e, \mathcal{N} \vDash \delta(\overline{b}, \overline{c})\right\} | \geq n$$

- Pretty obvious: þ-rank in $\mathbb{K}$ and in $Th(\mathfrak{M})$ coincide.
  - So, $\mathbb{K}$ is rosy if and only if $\mathfrak{M}$ is rosy.

þ-Independence and abstract independence relations, I

Theorem (Onshuus, Ealy-Onshuus; Adler)

For a complete theory T, the following are equivalent:

1. *T is rosy.*
2. *þ-Independence, $\mathop{\smile\!\!\!\!\!\!|}^{þ}$, is an indep. relation in models of T.*
3. *T admits some indep. relation with local character*
4. *T admits some indep. relation with symmetry and full transitivity.*

þ-Independence and abstract independence relations, II

Theorem

For a class $\mathbb{K}$, the following are equivalent:

1. $\mathbb{K}$ *is rosy.*
2. *þ-Independence,* $\mathop{\smile\!\!\!\!\!\!|}^{þ}$*, is an indep. relation in members of* $\mathbb{K}$*.*
3. $\mathbb{K}$ *admits some indep. relation with symmetry and full transitivity.*

- Here, independence relations only accommodates triples of finite sets.
- So, $3 \Rightarrow 1$ requires a trick in lifting to $\mathfrak{M}$.

Lifts of finitary independence relations

Given $\downfree$, a finitary independence relation, $A, B, C \subseteq \mathfrak{M}$, define $A \hat{\downfree}_C B$ to mean,

> there is a map $C_0 : A^{<\omega} \to C^{<\omega}$ such that for all $\overline{a} \in A^{<\omega}$, $\overline{b} \in B^{<\omega}$ and finite $D \subseteq C$, if $C_0(\overline{a}) \subseteq D$, then $\overline{a} \downfree_D \overline{b}$.

- This doesn't quite work – it can fail to have Existence, for example $(\forall A, C : A \hat{\downfree}_C C)$

Finitely-based and f.b.-rosy types

For $A, C \subset \mathfrak{M}$,

- $tp(A/C)$ is finitely-based if there is a finite $C_0 \subseteq C$ such that

  $$\overline{a} \downarrow_{C_0} D \text{ for all } \overline{a} \in A^{<\omega} \text{ and finite } C_0 \subseteq D \subseteq C$$

- $tp(A/C)$ is f.b.-rosy if for any $C \subseteq D \subseteq \mathfrak{M}$ such that $tp(D_0/C)$ is finitely-based for every finite $D_0 \subset D \setminus C$, there is a subset $C' \subseteq D$ such that $|C'| < (\aleph_0 + |A|)^+$ and $tp(A/D)$ does not þ-fork over C'.

Finitely-based types – closure properties

1. If $tp(A/B)$ is f.b. and $\sigma \in Aut(\mathcal{M})$, then $tp(\sigma A/\sigma B)$ is f.b.
2. If $A, B \subset \mathfrak{M}$ are finite, then $tp(A/B)$ f.b.
3. If $tp(A/B)$ is f.b. and $A_0 \subseteq A$, then $tp(A_0/B)$ is f.b.
4. If $tp(A/C)$ is f.b. and $A \hat{\downarrow}_C B$, then $tp(A/BC)$ is f.b.

From finitary independence to rosiness

Theorem

Suppose X is a set of types satisfying 1-4 of the previous slide with respect to a "notion of independence" $\downarrow$. Suppose

1. *$\downarrow$ is fully transitive for all triples:* $A \downarrow_C B_1 B_2 \Leftrightarrow A \downarrow_C B_1 \wedge A \downarrow_{CB_1} B_2$.
2. *If $tp(A/C) \in X$ and $tp(B/C) \in X$, then $A \downarrow_C B \Leftrightarrow B \downarrow_C A$*

Then every type in X is X-rosy.

Corollary

If $\mathop{\smile\!\!\!\!\!\mid}$ is a finitary independence relation in $\mathbb{K}$, then $\hat{\mathop{\smile\!\!\!\!\!\mid}}$ symmetric and transitive for finitely-based types, and every finitely-based type is f.b.-rosy. In particular, $\mathbb{K}$ is rosy.

$\mathbb{K}_\forall \to \mathbb{K}$

- Problem:
  "Given finite $A \leq \mathfrak{M}$, compute $\mathcal{B} \in \mathbb{K}$ with $A \leq \mathcal{B}$."
- This problem becomes interesting when:
  - We impose resource bounds on the program.
    (Hard to formulate)
  - We restrict the model of computation.

Inflationary fixed-points

- $\varphi(x_1 \ldots x_n; R^{(n)})$ a first-order formula, $\mathcal{A}$ a structure.
  $\varphi^0[\mathcal{A}] = \emptyset$
  $\varphi^{s+1}[\mathcal{A}] = \varphi^s[\mathcal{A}] \cup \{\bar{a} \in A^n : (\mathcal{A}, \varphi^s[\mathcal{A}]) \vDash \varphi(\bar{a})\}$
- $\varphi^\infty[\mathcal{A}] = \bigcup_s \varphi^s[\mathcal{A}]$

- Example: In the signature of graphs $\{E^{(2)}\}$, let

$$\varphi(x, y; R) = E(x, y) \vee \exists z(R(x, z) \wedge E(z, y))$$

Then $\varphi^\infty[G]$ is the transitive closure of the edge relation of G.

Efficient constructibility

- Given $A \leq \mathfrak{M}$ finite:
  1. Compute $A_i' = (A_i, \varphi_1^\infty[A_i], ..., \varphi_m^\infty[A_i])$;
  2. From a first-order test of A_i', choose,
     - a 0-definable set $D \subseteq A_i^n$ (in the sense of A_i');
     - a 0-definable equivalence relation $E \subseteq \mathfrak{M}^n \times \mathfrak{M}^n$.
  3. Set $A_{i+1} = acl(A_i \cup \pi_E[D])$.
- Repeat until $A_i \models Th^k(\mathfrak{M})$.

- Efficiently constructible $= \cdots$

  $\cdots =$ for every finite $A \leq \mathfrak{M}$, a model $\mathcal{M} \in \mathbb{K}$ with $A \leq \mathcal{M} \prec^k \mathfrak{M}$ is uniformly "close-to-definable" over A.

Rosiness from efficient constructibility

- One can define an independence relation in $\mathbb{K}$ by tracing through runs of the program.
- $\approx A \downarrow_C B$ if for any finite $BC \subseteq D \subset_{\text{fin}} \mathfrak{M}$, there is an $A' \equiv_{BC} A$ such that

  "C mediates all interaction between A' and D in a run of the program on $A' \cup D$."

Theorem

If $\mathbb{K}$ is efficiently constructible, then $\mathfrak{M}$ is rosy.