Proof of the Kim-Chernikov-Kaplan Lemma

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Motivation

NTP_2 = natural common generalisation of simplicity and NIP.

Morally it means: every type has bounded weight.

Chernikov and Kaplan recently proved an NTP_2 version of Kim's Lemma.
In this talk I will explain a slightly simplified version of their proof.

- Artem Chernikov and Itay Kaplan: 'Forking and dividing in NTP_2 theories'. J. Symbolic Logic 77 (2012), 1–20.
- Extension bases for weak invariance replaced by models.
- Strong splitting (or weak invariance) replaced by splitting (or invariance).
- Broom Lemma replaced by Hoover Lemma.

Kim's Lemma

Theorem (Kim)

Let T be simple.
For any $\varphi(x, b)$ and any C the following are equivalent.

1. $\varphi(x, b)$ *divides over* C.
2. $\varphi(x, b)$ *forks over* C.
3. *Every Morley sequence in* $\mathrm{tp}(b/C)$ *witnesses that* $\varphi(x, b)$ *divides over* C.
4. *Some Morley sequence in* $\mathrm{tp}(b/C)$ *witnesses that* $\varphi(x, b)$ *divides over* C.

Kim's Lemma for NTP_2 theories

Theorem (Chernikov, Kaplan)

Let T be NTP_2.
For any $\varphi(x, b)$ and any M the following are equivalent.

1. $\varphi(x, b)$ *divides over* M.
2. $\varphi(x, b)$ *forks over* M.
3. *Every strict Morley sequence in* $\text{tp}(b/M)$ *witnesses that* $\varphi(x, b)$ *divides over* M.
4. *Some strict Morley sequence in* $\text{tp}(b/M)$ *witnesses that* $\varphi(x, b)$ *divides over* M.

Outline

- Definitions
- Proof sketch:
  - $4 \implies 1 \implies 2$ is obvious.
  - Lemmas 1 and 2
    - Proof of Lemma 2
    - Lemma 2 says that $1 \implies 3$; $2 \implies 3$ is a simple corollary
    - Skipped proof of Lemma 1 is similar
  - Existence Lemma
    - Skipped proof uses Hoover Lemma
    - Implies $3 \implies 4$
  - Hoover Lemma
    - Proof of Hoover Lemma (uses Lemma 1)

Definition of TP_2

$\varphi(x, y)$ has TP_2 if a matrix of instances $\varphi(x, b)$ exists as follows.

$$\begin{array}{cccc} \varphi(x, b_{00}) & \varphi(x, b_{01}) & \varphi(x, b_{02}) & \cdots \\ \varphi(x, b_{10}) & \varphi(x, b_{11}) & \varphi(x, b_{12}) & \cdots \\ \varphi(x, b_{20}) & \varphi(x, b_{21}) & \varphi(x, b_{22}) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

- For some $k < \omega$, each row is k-inconsistent.
- For every $f : \omega \to \omega$,
  $\{\varphi(x, b_{i,f(i)}) \mid i < \omega\}$ is consistent.

(Fact: If such an array exists, then we can make the rows mutually indiscernible and the sequence of rows indiscernible.)

More definitions

$a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}_C B \iff \mathrm{tp}(a/BC)$ does not fork over C.

$a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_C B \iff \mathrm{tp}(a/BC)$ has a C-invariant global extension.

$(a_i)_{i<\omega}$ is an $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}$-Morley sequence over C if $a_i \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_C a_{<i}$ for all i.
(I.e. generated by a C-invariant global type.)

A global type $p(x)$ is *strictly invariant* over C if
$\forall B \supseteq C \; \forall a \models p{\upharpoonright}B$:

$$a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_C B \text{ and } B \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}_C a.$$

Strict Morley sequence over C:
generated by a strictly C-invariant global type.

Lemmas 1 and 2

Suppose $\varphi(x, b)$ is NTP_2 and divides over M.

Lemma 1

There is an $\mathop{\smile\hspace{-0.9em}\mid}\nolimits^{i}$*-Morley sequence over* M
which witnesses that $\varphi(x, b)$ *divides over* M.

Lemma 2

Let $q(y) \supset \mathrm{tp}(b/M)$ *be a strictly invariant global extension.*
Then every strict Morley sequence generated by q *over* M
witnesses that $\varphi(x, b)$ *divides over* M.

Proof of Lemma 2

Choose any M-indiscernible sequence $\bar{b}_0 = (b_{0i})_{i<\omega}$ witnessing that $\varphi(x, b)$ divides over M.
We may choose $\bar{b}_0$ so that $b \models q{\restriction}M\bar{b}_0$.
Using $\bar{b}_0 \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_M b$, we can find an $M\bar{b}_0$-indiscernible sequence $\bar{b}_1 \equiv_M \bar{b}_0$ in $\mathrm{tp}(b/M\bar{b}_0) = q{\restriction}M\bar{b}_0$.
We may also assume $b \models q{\restriction}\bar{b}_0\bar{b}_1$.
Continuing in this way, we get a matrix

$$\begin{matrix} b_{00} & b_{01} & b_{02} & \cdots \\ b_{10} & b_{11} & b_{12} & \cdots \\ b_{20} & b_{21} & b_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

such that for each row the φ-instances are k-inconsistent.
All vertical paths are generated by q and so have the same type.
By NTP_2 the φ-instances on vertical paths cannot all be consistent, so they are inconsistent. □

Existence Lemma

Lemma

Let T be NTP_2.
Every type over M has a strictly invariant global extension.
In other words:
In every type over M there is a strict Morley sequence.

We won't do the proof. It is straightforward once you know that a global type invariant over M does not fork over M.

. . . which is obvious.

Except that we need it for partial global types, in which case it's surprisingly hard to prove.

Hoover^{TM} Lemma

T any complete consistent theory.

Lemma

Let $p(x)$ be a partial global type, invariant over M.
Suppose $p(x) \vdash \psi(x, b) \vee \bigvee_{i<n} \varphi^i(x, c)$,
where $b \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}_M c$ and each $\varphi^i(x, c)$ divides over M.
Then $p(x) \vdash \psi(x, b)$.

Corollary

A consistent partial global type that is invariant over M does not fork over M.

Proof of corollary:
Let $p(x)$ be a partial global type invariant over M.
If p forks over M, then $p(x) \vdash \bot \vee \bigvee_{i<n} \varphi^i(x, c)$,
where each $\varphi^i(x, c)$ divides over M. Note that $\emptyset \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}_M c$.
By the lemma, $p(x) \vdash \bot$. $\square$

Proof of the Hoover Lemma (1)

Induction on n. Statement is trivial for $n = 0$.

Suppose it holds for n, and $p(x) \vdash \psi(x, b) \vee \bigvee_{i \leq n} \varphi^i(x, c)$,
where $b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_M c$ and each $\varphi^i(x, c)$ divides over M.
We must show: $p(x) \vdash \psi(x, b)$.

Let $(c_j)_{j<\omega}$ be an $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}$-Morley sequence over M,
witnessing that $\varphi^n(x, c)$ k-divides over M (some k).

$b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_M c \implies$ we may assume $b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_M (c_j)_{j<\omega}$
$\implies (c_j)_{j<\omega}$ is Mb-indiscernible.

By invariance of p:

$$p(x) \vdash \psi(x, b) \vee \bigwedge_{j<k'} \bigvee_{i \leq n} \varphi^i(x, c_j),$$

for every $k' < \omega$.

Proof of the Hoover Lemma (2)

$$p(x) \vdash \psi(x,b) \vee \bigwedge_{j<k'} \bigvee_{i\leq n} \varphi^i(x,c_j).$$

If we choose $k' = k$, then $\bigwedge_{j<k'} \varphi^n(x,c_j)$ is inconsistent and we get:

$$p(x) \vdash \psi(x,b) \vee \bigvee_{j<k'} \bigvee_{i<n} \varphi^i(x,c_j).$$

For each $j < k$ we have

1. $b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_{M} c_{\geq j} \implies b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_{Mc_{>j}} c_j$
2. $c_{>j} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_{M} c_j$.

$\implies bc_{>j} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_{M} c_j$ (by transitivity).

Since $bc_{>0} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_{M} c_0$, we can apply the induction hypothesis and get $p(x) \vdash \psi(x,b) \vee \bigvee_{1\leq j<k} \bigvee_{i<n} \varphi^i(x,c_j)$.
After eliminating $\bigvee_{i<n} \varphi^i(x,c_1)$ to $\bigvee_{i<n} \varphi^i(x,c_{k-1})$ in the same way, we get $p(x) \vdash \psi(x,b)$. $\square$

Postscript (2 February 2012):

For the present version I have removed most of the dynamic effects, corrected a number of typos and added a missing argument to the proof of Lemma 2.

As I said in the talk, the Hoover Lemma replaces a more complicated lemma of Chernikov and Kaplan, which they call the Broom Lemma as it is reminiscent of a sweeping operation. In the Hoover Lemma, unwanted formulas are sucked away one by one but other, more harmless formulas are added instead. Therefore I have dedicated the lemma to the Hoover-branded vacuum cleaner I had in Leeds, which required several passes to clean the carpet. (In the long run I will probably be more comfortable referring to it as the Vacuum Cleaner Lemma.)