Degenerate diffusion with nonlocal aggregation: behavior of radial solutions

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A degenerate diffusion equation with nonlocal drift

- We study the equation

$$u_t = \underbrace{\Delta u^m}_{\text{degenerate diffusion}} + \underbrace{\nabla \cdot (u \nabla (u * V))}_{\text{nonlocal aggregation}}, \qquad (P)$$

- This PDE describes biological aggregation:
  u – population density
  V – models the long-range attraction

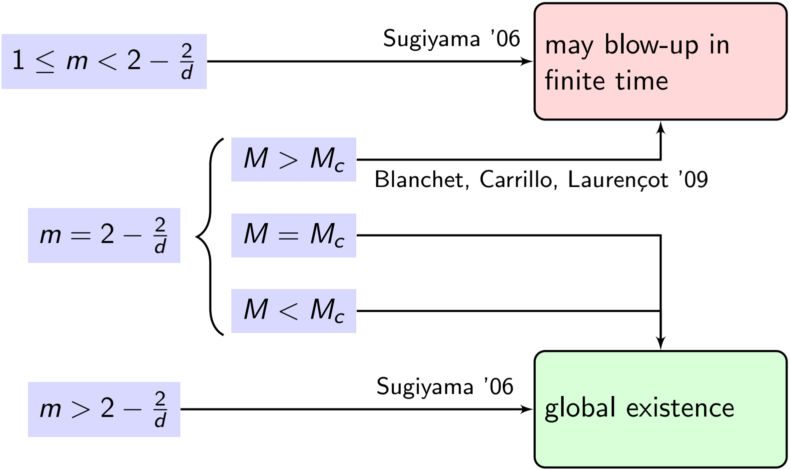
- We focus on the case when V is of power-law form, i.e. $V(x) = -\frac{1}{|x|^\gamma}$.

An aggregation equation with degenerate diffusion

- When $m = 1$ and V is Newtonian kernel, it becomes the well-known Patlak-Keller-Segel model.
- The degenerate diffusion term is introduced by Boi, Capasso, Morale (2000) and Topaz, Bertozzi, Lewis (2006) to avoid overcrowding.
- When V is the Newtonian kernel, $m = 2 - 2/d$ gives the exact balance between the diffusion term and aggregation term.

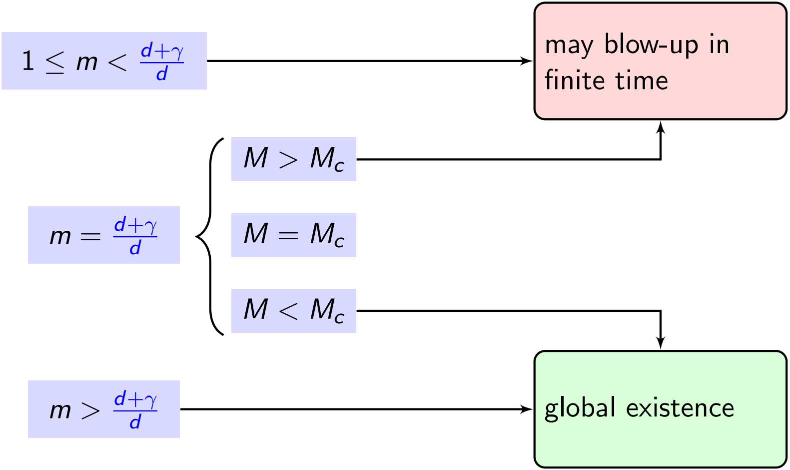
Global Well-posedness v.s. Finite-time Blow-up

When V is Newtonian potential $-\frac{1}{|x|^{d-2}}$:



Global Well-posedness v.s. Finite-time Blow-up

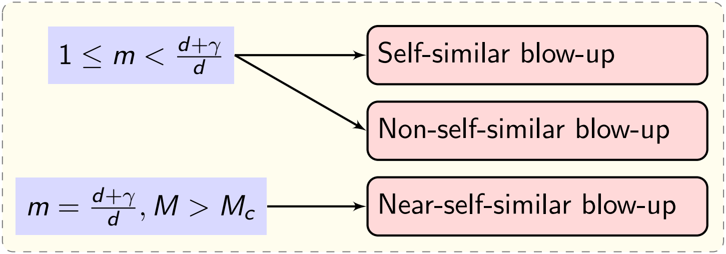
For a power-law kernel $V = -\frac{1}{|x|^\gamma}$: (Bedrossian-Rodríguez-Bertozzi,'10)



Numerical Method and Blow-up Behaviors

We use an arbitrary Lagrangian Eulerian method with adaptive mesh refinement.

- Aggregation step: we adopt the method by Huang and Bertozzi (2010) and let the mesh move with the particles.
- Diffusion step: we use an implicit finite volume scheme to solve the degenerate diffusion equation on a fixed mesh.



$m < (d+\gamma)/d$: Self-similar Blow-up

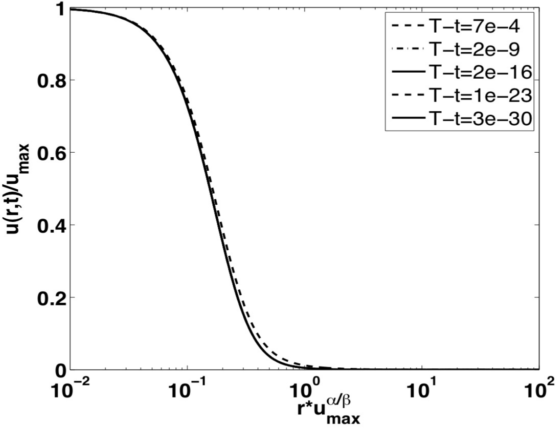
(movie of the log-log plot of the density)

As t goes to the blow-up time T, no mass is concentrating at the origin.

$m < (d+\gamma)/d$: Self-similar Blow-up

The scaling of the solution $u(x,t)$ is

$$u(x,t) \sim (T-t)^{-\beta} w\left(\frac{x}{(T-t)^{\alpha}}\right) \text{ as } t \to T$$



$m < (d+\gamma)/d$: Non-self-similar Blow-up

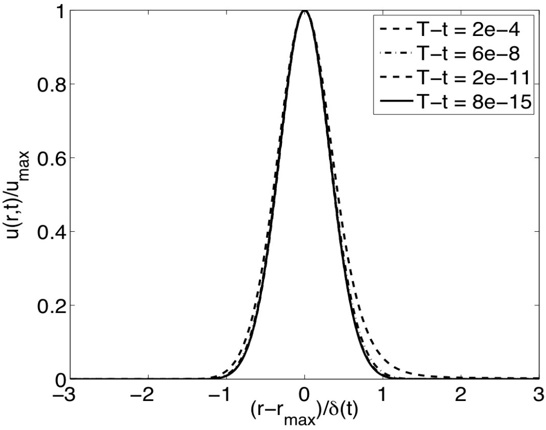
(movie of the log-log plot of the density)

As $t \to T$, the peak contains a finite amount of mass.

$m < (d+\gamma)/d$: Non-self-similar Blow-up

The blow-up profile for $u(r,t)$ is $u(r,t) \sim Q(t)\,\varphi\left(\frac{r-R(t)}{\delta(t)}\right)$,

where $Q(t) \to \infty$, $\delta(t) \ll R(t) \to 0$ as $t \to T$.



$m < (d+\gamma)/d$: What happens in between?

We carefully adjust the initial data to get a separatrix between a self-similar and non-self-similar blow-up:

(movie of the log-log plot of the density)

Brenner-Constantin-Kadanoff-Schenkel-Venkataramani (1999) observed similar behavior when $m = 1$ and K is Newtonian kernel.

(movie of the log-log plot of the density)

As $t \to T$, the peak area contains exactly the critical mass M_c.

$m = \frac{d+\gamma}{d}$: Near-self-similar Blow-up

- The scaling of the solution $u(x,t)$ is

$$u(r,t) = \frac{1}{R(t)^d}\bar{u}(\frac{r}{R(t)}) + 1_{\{r>R(t)\}}f(r),$$

- Here $\bar{u}$ is the stationary solution for $M = M_c$.
- $R(t) \sim (T-t)^\alpha g(T-t)$, where $g(T-t)$ is some logarithmic correction term. When $m = 1$, the correction term is computed by Herrero-Velazquez (1996) using matched asymptotic methods.

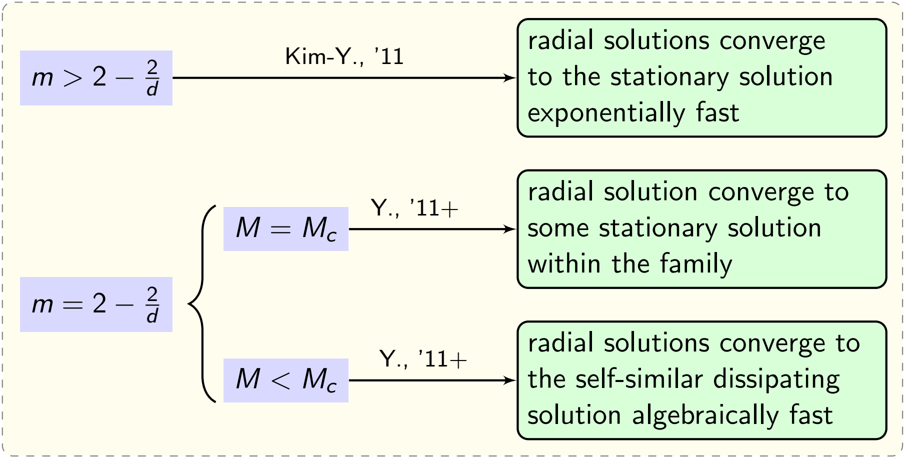
Asymptotic behavior for global solutions: previous results

From now on we assume that V is the Newtonian potential.

$m > 2 - \frac{2}{d}$	Existence: Lions '84; Uniqueness: Lieb-Yau '87	there exists a unique radial stationary solution for every mass size
$m = 2 - \frac{2}{d}$, $M = M_c$	Blanchet-Carrillo-Laurençot '09	there is a family of stationary solutions which are scalings of each other
$m = 2 - \frac{2}{d}$, $M < M_c$	Blanchet-Carrillo-Laurençot '09	there exists a self-similar dissipating solution

- However the asymptotic behavior of solutions was not clear.

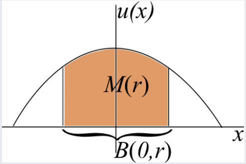
Asymptotic behavior for radial solutions



Our main tool: mass comparison

Definition

Given two radially symmetric non-negative function u_1 and u_2, we define



$$M_i(r) := \int_{B(0,r)} u_i(y)dy.$$

We say u_1 *is less concentrated than* u_2, or $u_1 \prec u_2$ if for any $r > 0$, we have $M_1(r) \leq M_2(r)$.

- When V is the Newtonian potential, $u_1(\cdot, 0) \prec u_2(\cdot, 0) \Longrightarrow u_1(\cdot, t) \prec u_2(\cdot, t)$.
- This enables us to construct subsolutions and supersolutions in the mass comparison sense.

Difficulties in nonradial solutions

- In Kim-Y. '11, for Newtonian potential V, we showed that $\|u(\cdot,t)\|_p \leq \|\bar{u}(\cdot,t)\|_p$ for any time t, where $\bar{u}$ is the solution to (P) with a symmetrized initial data $u^*(\cdot,0)$.
- What about the other direction?
- For $m > 2 - 2/d$ and Newtonian potential V,
  - Does there exist any non-radial stationary solutions?
  - Does a non-radial solution with compactly supported initial data stay in some compact set forever?

Thank you for your attention!