On the propagation of chaos in pair interaction driven master equations

Bernt Wennberg
Mathematical Sciences
Chalmers
in collaboration with Eric Carlen, Robin Chatelin and Pierre Degond

Plan of the talk

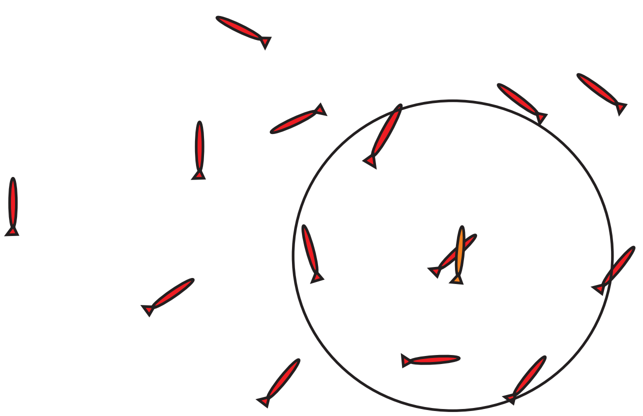
- The Vicsek model and the BDG-Boltzmann equation
- Propagation of chaos
- The "Choose the leader model"
- More about BDG
- Simulation results

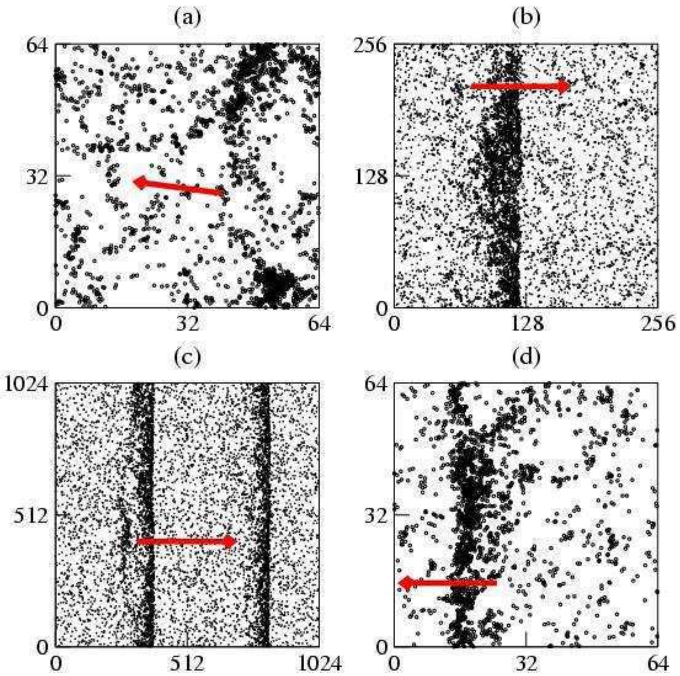
The Vicsek model

"The motion of self-propelled particles"

$$v_i(t+\Delta t) = v_0 \theta \left[\sum_{j \in S_i} v_j(t) + \eta \mathcal{N}_i \xi \right]$$

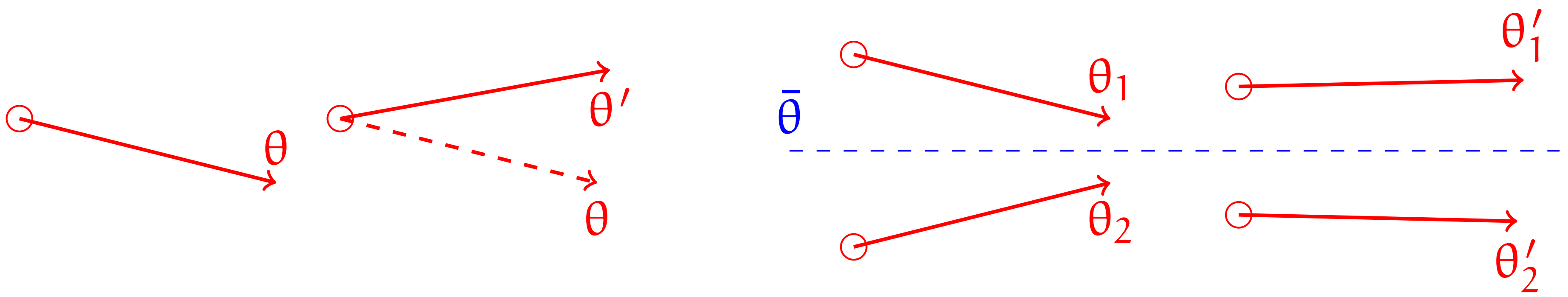
$$v_i(t+\Delta t) = v_0 \left(\mathcal{R}_\eta \circ \theta \right) \left[\sum_{j \in S_i} v_j(t) \right]$$





Chaté, Ginelli, Grégoire, Raynaud, Collective motion of self-propelled particles interacting without cohesion, arXiv, dec 2007

The Vicsek model: A Boltzmann equation



$$\frac{\partial f}{\partial t}(r,\theta,t) + e(\theta)\cdot\nabla f(x,\theta,t) =$$

$$-\lambda f(r,\theta,t) + \lambda \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} p_0(\theta-\eta-\theta')f(r,\theta',t)\,d\eta\,d\theta'$$

$$-f(r,\theta,t)\int_{-\pi}^{\pi} |e(\theta')-e(\theta)| f(r,\theta,t)\,d\theta'$$

$$+\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} p(\theta-\bar{\theta}-\eta)|e(\theta_2)-e(\theta_1)|f(r,\theta_1,t)f(r,\theta_2,t)\,d\eta\,d\theta_1\,d\theta_2$$

E. Bertin, M. Droz, G. Grégoire

The BDG-Boltzmann equation

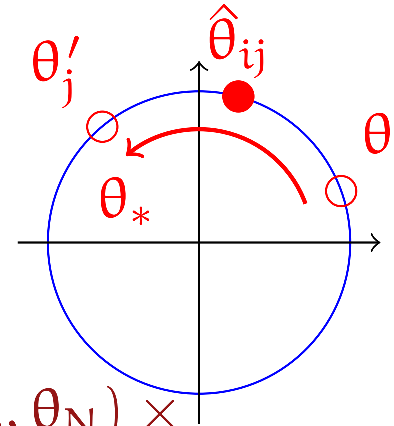
- Bertin et al: show that their binary model qualitatively similar to Vicsek
- – derive fluid equations by “Fourier closure”
- What about closure via “Maxwellian”?
  What is then the Maxwellian?

The BDG-Boltzmann equation

- Bertin et al: show that their binary model qualitatively similar to Vicsek
- – derive fluid equations by "Fourier closure"
- What about closure via "Maxwellian"?
  What is then the Maxwellian?
- Can the BDG-equation be rigorously derived from a many particle system?
- The homogeneious BDG-Boltzmann equation

$$\partial_t f(t,\theta) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big(f(t,\theta')f(t,\theta'+\theta_*)g(\theta-\theta'-\frac{\theta_*}{2}) \\ -f(t,\theta)f(t,\theta+\theta_*)\Big)\beta(|\sin(\theta_*/2)|)\frac{d\theta'}{2\pi}\frac{d\theta_*}{2\pi}$$

A particle system for the BDG model



- The adjoint Markov transition operator

$$Q_N^* f_N(\theta_1, ..., \theta_N) =$$
$$\frac{2}{N(N-1)} \sum_{i<j} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f_N(\theta_1, ..., \theta_i', ..., \underbrace{\theta_i' + \theta^*}_{\theta_j'}, ..., \theta_N) \times$$
$$g(\theta_i - \theta_i' - \frac{\theta^*}{2}) g(\theta_j - \theta_i' - \frac{\theta^*}{2}) \frac{d\theta_i'}{2\pi} \frac{d\theta^*}{2\pi}$$

- The master equation

$$\begin{aligned} f_N &= f_N(t, \theta_1, ..., \theta_N) \\ \frac{\partial}{\partial t} f_N(t, v_1, ..., v_N) &= N\left(Q_N^* - I\right) f_N(t, \theta_1, ..., \theta_N) \end{aligned}$$

The BDG model

- Marginals:

$$f_N^{(k)}(t,\theta_1,...,\theta_k) \;=\; \int\cdots\int f_N(t,\theta_1,...,\theta_k,\theta_{k+1},...,\theta_N)\, d\theta_{k+1}\cdots d\theta_N$$

- Evolution of marginals:

$$\frac{\partial}{\partial t} f_N^{(k)}(t,\theta_1,...,\theta_k) = \frac{k(k-1)}{N-1} Q_k^* f_N^{(k)}(t,\theta_1,...,\theta_k) + \frac{2(N-k)}{N-1}\sum_{i\leq k}\iint f_N^{(k+1)}(t,\theta_1,...,\theta_i',...,\theta_k,\theta_i'+\theta_*)\times g(\theta_i-\theta_i'-\frac{\theta^*}{2})g(\theta_j-\theta_i'-\frac{\theta^*}{2})\,\frac{d\theta_i'}{2\pi}\frac{d\theta^*}{2\pi} - -\frac{2N-3k+k^2}{N-1} f_N^{(k)}(t,\theta_1,...,\theta_k)$$

The BDG model

- Formally, when $N \to \infty$

$$\begin{aligned}\partial_t f^{(1)}(t,\theta) \\ &= 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f^{(2)}(t,\theta',\theta'+y)g(\theta_1-\theta_1'-\frac{\theta_*}{2})\frac{d\theta_1'}{2\pi}\frac{d\theta_*}{2\pi} \\ &\quad -2f^{(1)}(t,\theta)\end{aligned}$$

The BDG model

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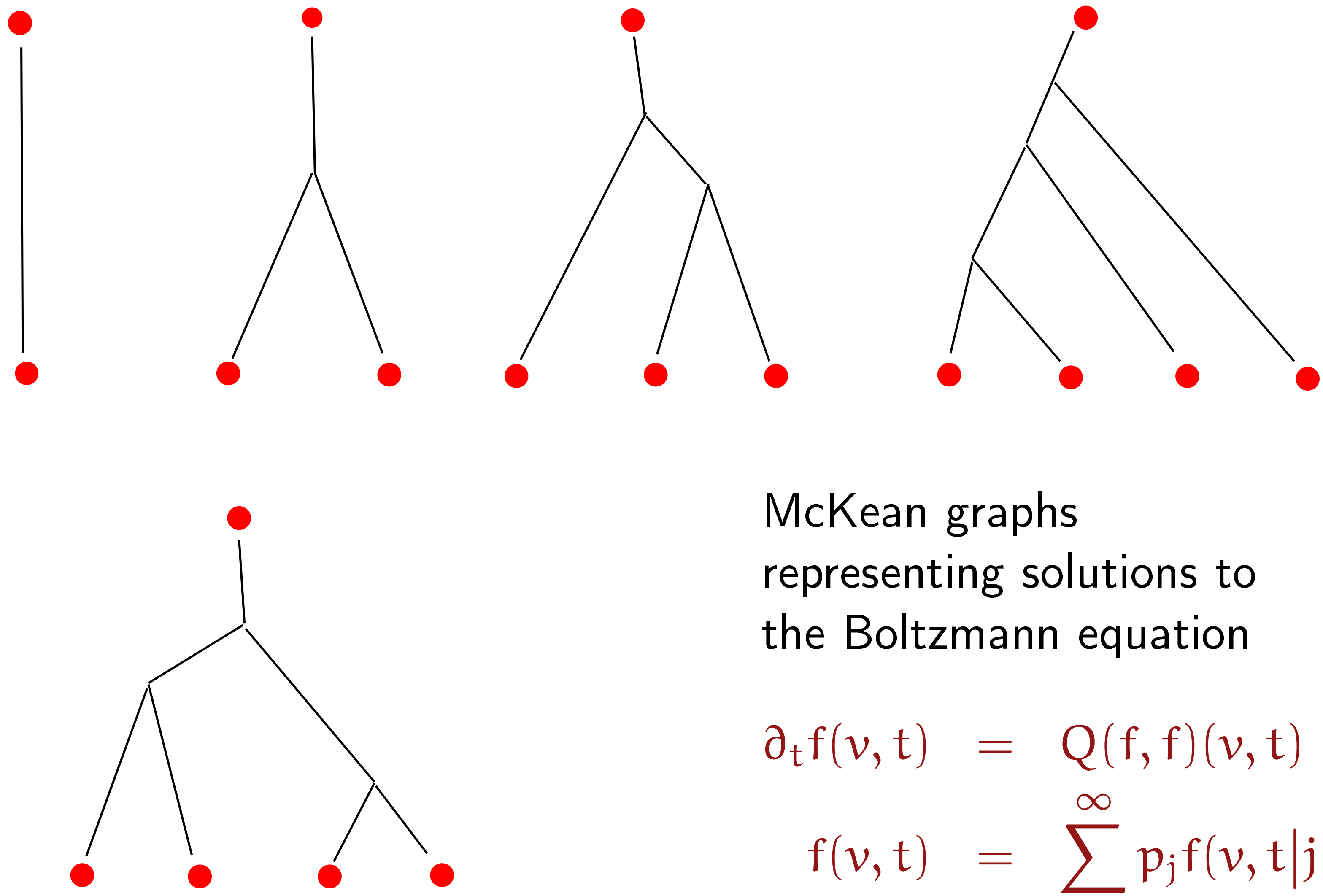
&\quad -2f^{(1)}(t,\theta)
\end{aligned}
$$

- The chaos assumption: $f^{(2)}(t,\theta_1,\theta_2) = f^{(1)}(t,\theta_1)f^{(1)}(t,\theta_2)$

$$
\begin{aligned}
\partial_t f(t,\theta) &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t,\theta')f(t,\theta'+\theta_*)g(\theta-\theta'-\frac{\theta_*}{2})\frac{d\theta'}{2\pi}\frac{d\theta_*}{2\pi} \\
&\quad -f(t,\theta)
\end{aligned}
$$

Propagation of chaos

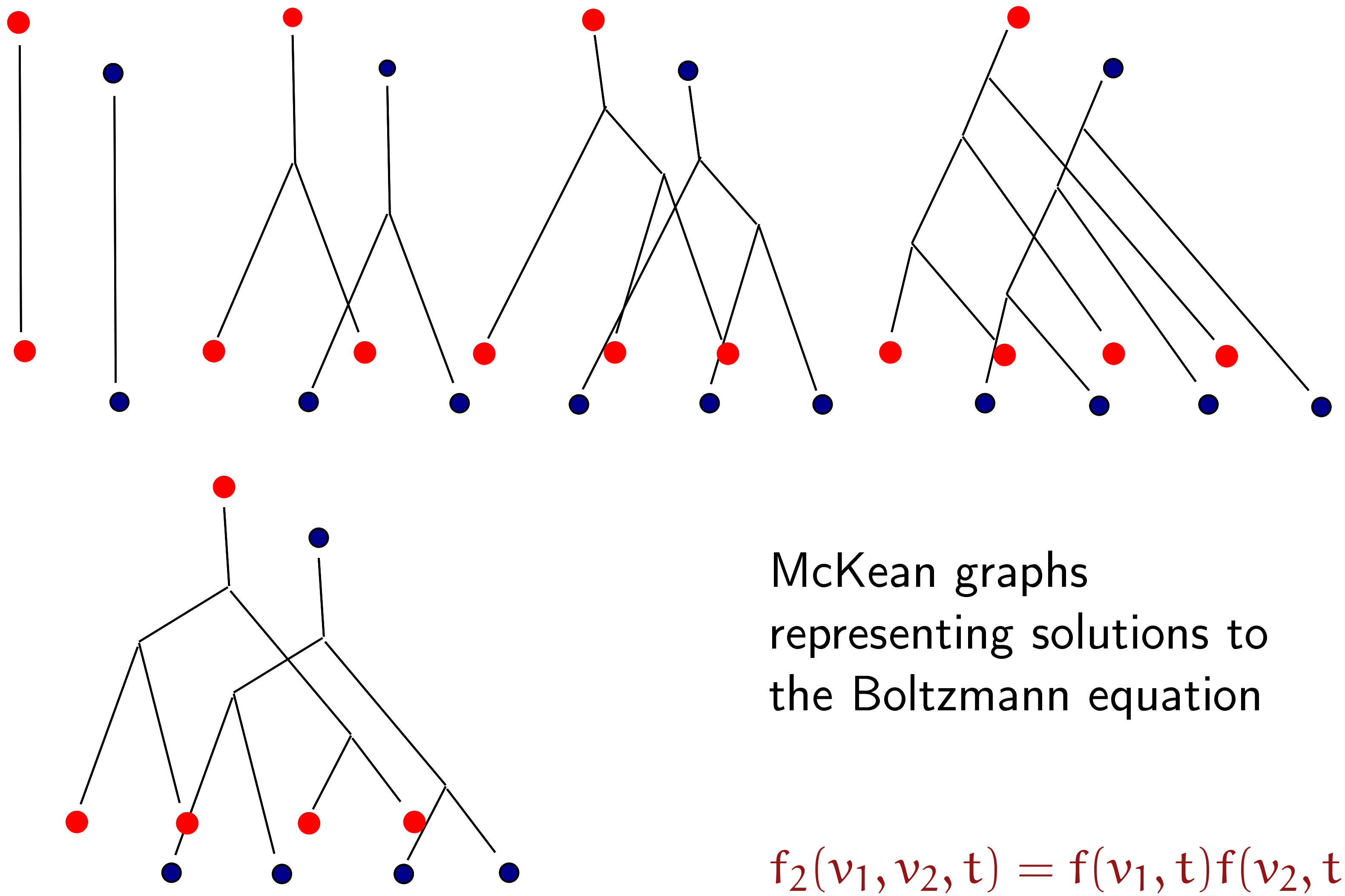


McKean graphs representing solutions to the Boltzmann equation

$$\partial_t f(v,t) = Q(f,f)(v,t)$$

$$f(v,t) = \sum_{j=0}^{\infty} p_j f(v,t \,|\, j)$$

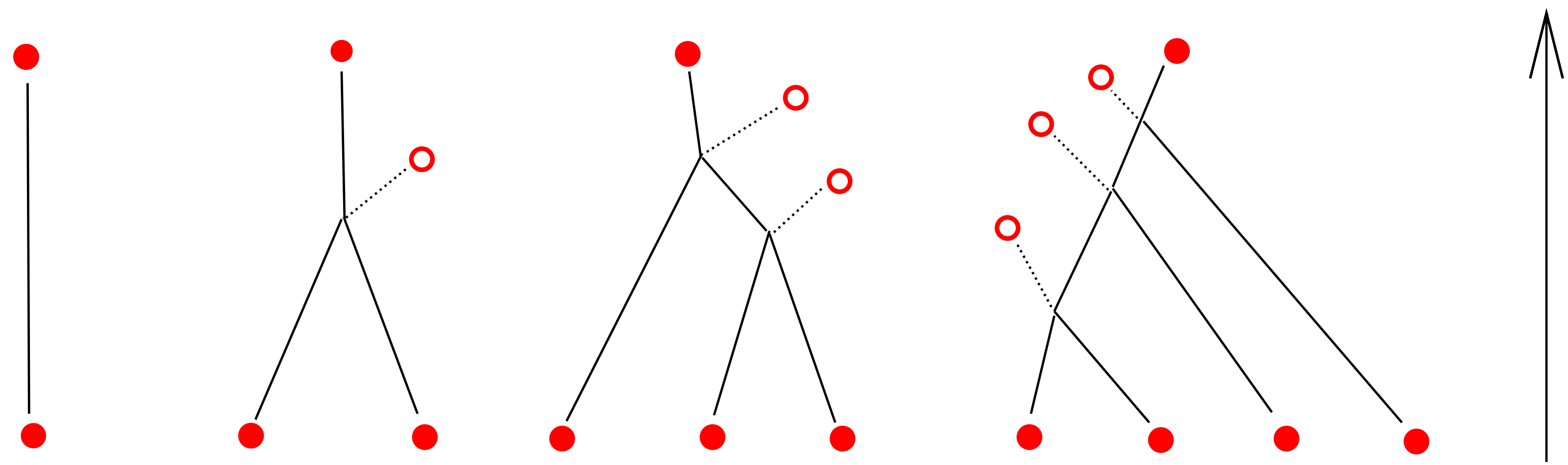
Propagation of chaos

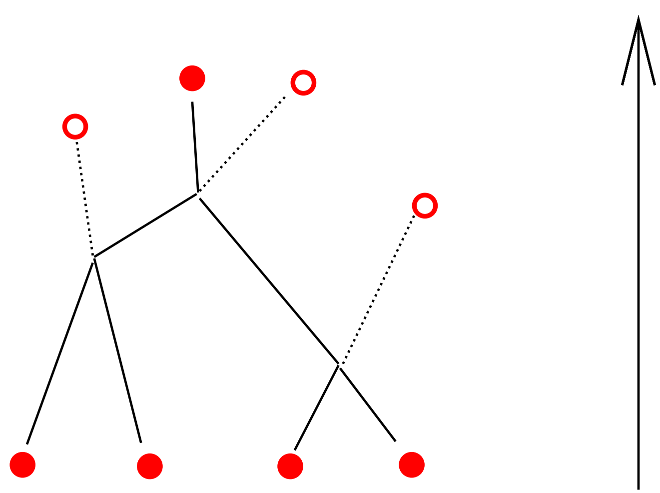


McKean graphs representing solutions to the Boltzmann equation

$$f_2(v_1, v_2, t) = f(v_1, t) f(v_2, t) \quad ??$$

Propagation of chaos





Finitely many particles
$\Longrightarrow$ positive probability
of recollisions.

Propagation of chaos:
this does not happen with
infinitely many particles.

Propagation of chaos according to Kac

Definition: A sequence of probability measures $f_N(v_1, ..., v_N)$, $N = 1, ..., \infty$ is said to have the Boltzmann property, or to be chaotic if for each k,

$$\lim_{n\to\infty} f_N^{(k)}(v_1, ..., v_k) \to \prod_{j=1}^{k} \lim_{n\to\infty} f_N^{(1)}(v_j, t)$$

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Theorem[M. Kac] Propagation of chaos holds for a particular model of the Boltzmann equation.

Pair interaction driven master equations

- $V = (v_1, ..., v_N) \in E^N$
- Poisson stream of jump times t_j, at which
$$(v_i, v_j) \to (v_i', v_j') = (W_i(v_i, v_j), W_j(v_i, v_j))$$
- V_k state after jump nr k
- Markov transition operator:
For $\phi \in C(E^N)$, define $Q\phi(\mathbf{v}) = E\left[\phi(V_{k+1}) \,\middle|\, V_k = \mathbf{v}\right]$.
- Let $F_k(\mathbf{v})$ be probability density of V_k.
- Adjoint of Markov transition operator:
$$\int_{E^N} \phi(\mathbf{v}) F_{k+1}(\mathbf{v})\, d^N v = \int_{E^N} Q\phi(\mathbf{v}) F_k(\mathbf{v})\, d^N v = \int_{E^N} \phi(\mathbf{v}) Q^* F_k(\mathbf{v})\, d^N v$$

Definition

A PAIR INTERACTION DRIVEN MASTER EQUATION is an equation of the form

$$\frac{\partial}{\partial t}F(\mathbf{v},t) = L^{*}F(\mathbf{v},t)$$

where

- F is a probability density on E^N
- $$L^{*} = \frac{N}{2}\sum_{i<j} p_{i,j}(\mathbf{v})\left(Q^{*}_{i,j} - I\right),$$
- $Q_{(i,j)}$ a Markov transition operator on E^N
- $p_{i,j}$ are pair selection probabilities: $\sum_{i<j} p_{i,j}(\mathbf{v}) = 1$.

From now on, $p_{i,j} = \dfrac{2}{N(N-1)}$

(almost) Kac's approach

- $\int_{E^N} F(\mathbf{v},t)\phi(\mathbf{v})\,dv_1,\ldots,dv_N = \int_{E^N} F(\mathbf{v})e^{tL_N}\phi(\mathbf{v})\,dv_1,\ldots,dv_N.$
- Studying a k-th marignal equivalent to taking $\phi = \phi(v_1,\ldots,v_k)$.
- Power series of e^{tL_N} convergent "uniformly in N".
- Analyse term by term and prove *e.g.*
- $$\lim_{N\to\infty}\int_{E^N} F_{0,N}e^{tL}\phi_1(v_1)\phi_2(v_2)\,dv_1\ldots,dv_N =$$ $$\left(\lim_{N\to\infty}\int_{E^N} F_{0,N}e^{tL}\phi_1(v_1)\,dv_1\ldots,dv_N\right)\left(\lim_{N\to\infty}\int_{E^N} F_{0,N}e^{tL}\phi_2(v_2)\,dv_1\ldots,dv_N\right)$$

Estimates on $\exp(tL)$

THEOREM: *Let* $\phi = \phi(v_1, ..., v_K)$*, and for* $N > K$*, let* $(v_1, ..., v_N) \mapsto \phi(v_1, ..., v_k)$*. Then there is* $t_0 > 0$ *such that*

$$e^{tL} = \sum_{k=0}^{\infty} \frac{t^k}{k!} L^k \phi$$

converges absolutely in L^∞*, uniformly in* N*, for* $0 \leq t < t_0$*.*

PROOF: If $i, j > K$, then $(Q_{(i,j)} - I)\phi = 0$, and hence

$$\frac{1}{N-1} \sum_{i<j} \left(Q_{(i,j)} - I\right) \phi \quad = \quad \frac{1}{N-1} \sum_{i=1}^{K} \sum_{j=i+1}^{N} \left(Q_{(i,j)} - I\right) \phi \qquad \Longrightarrow$$

$$\|L\phi\|_\infty \quad \leq \quad 2\frac{2}{N-1} \frac{K(2N-K-1)}{2} \|\phi\|_\infty \leq 2K\|\phi\|_\infty .$$

Estimates on $\exp(tL)$

- For $\phi = \phi(v_1)$, L^k involves $v_1, v_2, \ldots, v_{k+1}$

$$\Downarrow$$

$$\|L^k \phi\|_\infty \leq 2^k k! \|\phi\|_\infty$$

$$\Downarrow$$

$$\frac{t^k}{k!} \|L^k \phi\|_\infty \leq (2t)^k \|L^k \phi\|_\infty$$

$$\Downarrow$$

  Series convergent for $t < 1/2$

- Similar estimates for $\phi = \phi(v_1, \ldots, v_K)$.
- The symmetry assumption necessary for this estimate.

A combinatorial lemma

LEMMA *Let* $F_{0,N}$ *be symmetric,* $\phi^{(k)} = \phi^{(k)}(v_1, ..., v_k)$.
Define

$$\phi^{(k+1)}(v_1, ..., v_k, v_{k+1}) = \sum_{i=1}^{k} \left(Q_{(i,k+1)} - I\right) \phi^{(k)}(v_1, ..., v_k)$$

Then

$$\lim_{N\to\infty} \int_{E^N} F_{0,N} L\phi^{(k)} dv_1 \cdots dv_N = \lim_{N\to\infty} \int_{E^N} F_{0,N} \phi^{(k+1)} dv_1 \cdots dv_N$$

- Proof direct from definition
- It follows ...

$$\lim_{N\to\infty} \int_{E^N} F_{0,N} e^{tL} \phi^m dv_1, ..., dv_N =$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \lim_{N\to\infty} \int_{E^N} F_{0,N} \phi^{(m+k)} dv_1, ..., dv_N .$$

Propagation of chaos

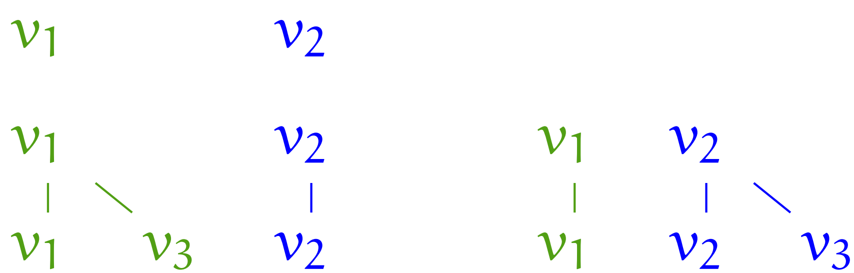
- For chaotic initial data

$$\lim_{N\to\infty}\int_{E^N} F_{0,N}\phi^{(m+k)}\,dv_1,\ldots,dv_N =$$
$$= \int_{E^{m+k}}\prod_{j=1}^{k+m} f(v_j)\phi^{(m+k)}(v_1,\ldots,v_{m+k})dv_1\ldots dv_{k+m}$$

- Can also prove

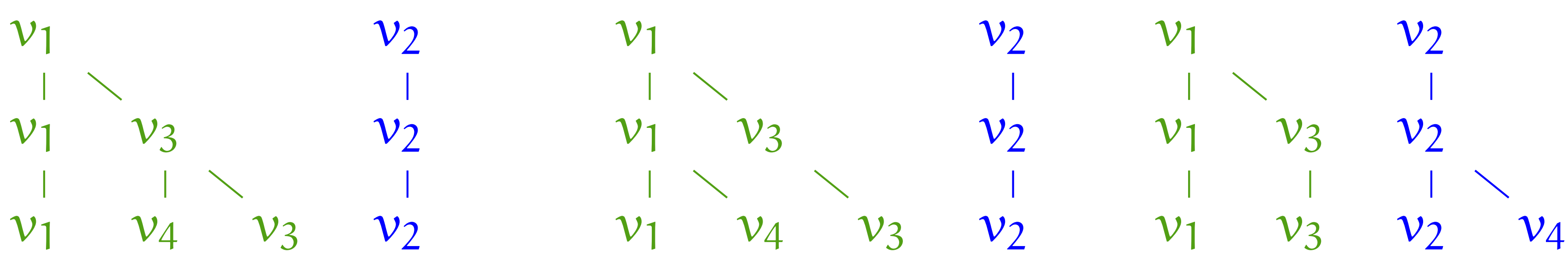
$$\lim_{N\to\infty}\int_{E^N} F_{0,N}e^{tL}\phi_1(v_1)\phi_2(v_2)\,dv_1\ldots,dv_N =$$
$$\left(\lim_{N\to\infty}\int_{E^N} F_{0,N}e^{tL}\phi_1(v_1)\,dv_1\ldots,dv_N\right)\left(\lim_{N\to\infty}\int_{E^N} F_{0,N}e^{tL}\phi_2(v_2)\,dv_1\ldots,dv_N\right)$$

Propagation of chaos



$\phi_1(v_1)\phi_2(v_2)$

$\phi_1^{(2)}(v_1, v_3)\phi_2(v_2)+$
$\phi_1(v_1)\phi_2^{(2)}(v_2, v_3)$



$$\lim_{N\to\infty}\int_{E^N} F_{0,N}e^{tL}\phi_1(v_1)\phi_2(v_2)\, dv_1..., dv_N =$$

$$\left(\lim_{N\to\infty}\int_{E^N} F_{0,N}e^{tL}\phi_1(v_1)\, dv_1..., dv_N\right)\left(\lim_{N\to\infty}\int_{E^N} F_{0,N}e^{tL}\phi_2(v_2)\, dv_1..., dv_N\right)$$

Propagation of chaos

THEOREM: *Let L be the generator of a pair interaction driven master equation, and $\{F_{0,N}\}$ be f-chaotic. Let $f(v,t)$ satisfy*

$$\frac{\partial}{\partial t} f(v_1,t) = 2\left(\int_E Q^*_{(1,2)} f^{\otimes 2}(v_1,v_2)\, dv_2 - f(v_1,t)\right)$$

Then $\left\{e^{tL^*} F_{0,N}\right\}$ is $f(\cdot,t)$-chaotic.

- The difference between this and Kac's theorem is that Q need not be reversible
- The BDG-model satisfies the hypotheses.

Another example: A choose the leader model

- Each animal (fish) moves with speed $v \in S^1$.
  v represented as $\theta \in [0, 2\pi[$ or $v = e^{i\theta} \in \mathbb{C}$.
- When two fish meet, one tries to choose the velocity of the other, but makes a random error
- $(\theta_i, \theta_j) \mapsto (\theta_j + \xi, \theta_j)$ or $(\theta_i, \theta_j) \mapsto (\theta_i, \theta_i + \xi)$
  where $\xi \in [-\pi, \pi[$ is random.

  alternatively

  $(v_i, v_j) \mapsto (Wv_j, v_j)$ or (v_i, Wv_i) with $W \in \mathbb{C}, |W| = 1$.
- Density for noise term: $g(z)$ (assume $g(z) = g(z^*)$)

A master equation for the Choose the leader model

- Markov transition operator:

$$Q\varphi(\mathbf{v}) = \frac{1}{N(N-1)} \sum_{i<j} \int_{S^1} \Big[\varphi(v_1, \dots, zv_j, \dots, v_j, \dots, v_N) + \varphi(v_1, \dots, v_i, \dots, zv_i, \dots, v_N) \Big] g(z)dz\ .$$

- $F_k(\mathbf{v})$ probability density of V_k, the state after jump k.

$$\int_{(S^1)^N} \varphi(\mathbf{v}) F_{k+1}(\mathbf{v}) d^N v = \int_{(S^1)^N} Q\varphi(\mathbf{v}) F_k(\mathbf{v}) d^N v\ .$$

- $$F_{k+1} = Q^* F_k$$

A master equation for the Choose the leader model

- Adjoint Markov transition operator

$$Q^*F(\mathbf{v}) = \frac{1}{N(N-1)} \sum_{i<j} \Big[[F_k]_{\hat{i}}(v_1, \dots, \hat{v_i}, \dots, v_N) + [F_k]_{\hat{j}}(v_1, \dots, \hat{v_j}, \dots, v_N) \Big] g(v_i^* v_j) \; .$$

- Marginal of F

$$[F_k]_{\hat{i}} = \int_{S^1} F_k(v_1, ..., v_i, ..., v_N) dv_i$$

- Master equation:

$$\frac{\partial}{\partial t} F(\mathbf{v}) = \underbrace{N \binom{N}{2}^{-1} \sum_{i<j} \left(Q^*_{(i,j)} - I \right)}_{L^*} F(\mathbf{v})$$

Invariant densities

- $$F_\infty(\vec{v}) = \frac{1}{N(N-1)} \sum_{i<j} \Big[[F_\infty]_{\hat{i}}(v_1, \ldots, \hat{v}_i, \ldots, v_N) + [F_\infty]_{\hat{j}}(v_1, \ldots, \hat{v}_j, \ldots, v_N) \Big] g(v_i^* v_j) \,.$$
- $1 \neq g(v_i^* v_j)$, hence uniform density not invariant
- Marginals:

$$F_\infty^{(1)}(v_1)$$
$$= \frac{1}{N(N-1)} \sum_{j=2}^{N} \int_{T_{N-1}} \Big[[F_\infty]_{\hat{1}}(\hat{v}_1, \ldots) + [F_\infty]_{\hat{j}}(\ldots, \hat{v}_j, \ldots) \Big] g(v_1^* v_j) dv_2 \cdots dv_N$$
$$+ \frac{N-2}{N} F_\infty^{(1)}(v_1)$$

Invariant densities

- .... $\Longrightarrow \quad F_\infty^{(1)}(v_1) = F_\infty^{(1)} * g(v_1) \quad \Longrightarrow \quad F_\infty^{(1)}$ is the uniform distribution.
- Marginals:

$$F_\infty^{(2)}(v_1, v_2) = \frac{1}{N-1} g(v_1^* v_2) + \frac{N-2}{2(N-1)} H(v_1, v_2)$$

where

$$H(v_1, v_2) = \int_{S^1} F_\infty^{(2)}(v_2, z) g(z^* v_1) dz + \int_{S^1} F_\infty^{(2)}(v_j, v_1) g(z^* v_2) dz$$

- Can be solved using Fourier transform: $F_\infty^{(2)}(v_1, v_2) = f(v_1^* v_2)$,

$$f = \frac{1}{N-2} \sum_{\ell=1}^{\infty} \left[\left(\frac{N-2}{N-1} \right)^{\ell} g^{*\ell} \right] .$$

Invariant densities

- If g is fixed, f becomes uniform as $N \to \infty$
- With $g = g_N$:

$$\widehat{f}(k) = \widehat{g}_N(k)\left[1 - (N-2)(\widehat{g}_N(k) - 1)\right]^{-1}$$

- Conclusion: CL-model too diffusive unless the noise is scaled with the number of particles (fish):
  $\partial_t F(v,t) = 0$
- With suitably scaled noise,
  $F^{(2)}_{\infty,N} \to f(v_1^* v_2)$
- The family of invariant densities is not chaotic
- Also the BDG-model has different behaviour

The BDG-model

-
$$\partial_t f(t,\theta) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t,\theta')f(t,\theta'+\theta_*)g(\theta-\theta'-\frac{\theta_*}{2})\frac{d\theta'}{2\pi}\frac{d\theta_*}{2\pi}$$
$$-f(t,\theta)$$

- Look for solution as a Fourier series:

$$f(t,\theta) = \sum_{k=-\infty}^{\infty} a_k(t)e^{ik\theta} \qquad a_k(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta} f(t,\theta)\,d\theta$$

-
$$\frac{da_k}{dt} = \sum_n a_{k-n}a_n\left(\gamma_k\Gamma(n-k/2)-\Gamma(k)\right)$$

Linearized stability of the uniform distribution

- $f(t,\theta) = 1 + \varepsilon \sum_{k=-\infty}^{\infty} b_k(t) e^{ik\theta}$
- $\frac{d}{dt} b_k(t) = b_k(t) \underbrace{\left(2\gamma_k \Gamma(k/2) - \Gamma(0) - \Gamma(k) \right)}_{\lambda_k} + \mathcal{O}(\varepsilon)$
- All modes for $k > 1$ are stable, and for $k = 1$ if

$$\lambda_1 = \gamma_1 2\Gamma(1/2) - \Gamma(0) - \Gamma(1) < 0 \qquad \Leftrightarrow \qquad \gamma_1 < \frac{\pi}{4}$$

- Example: $g_\tau(y) = 2\pi \sum_{j=-\infty}^{\infty} \frac{1}{\tau} \rho\left(\frac{y - 2\pi j}{\tau}\right) \quad \Rightarrow \quad \gamma_k = \hat{\rho}(\tau k)$

Non uniform stationary distributions (Maxwellian case)

- $a_k = \sum_n a_{k-n} a_n \gamma_k \frac{\sin(\pi(n-k/2))}{\pi(n-k/2)}$
- $$\gamma_k = \frac{1}{\sum_{n=-\infty}^{\infty} \frac{a_n a_{k-n}}{a_k} \frac{\sin(\pi(n-k/2))}{\pi(n-k/2)}} \quad (1)$$

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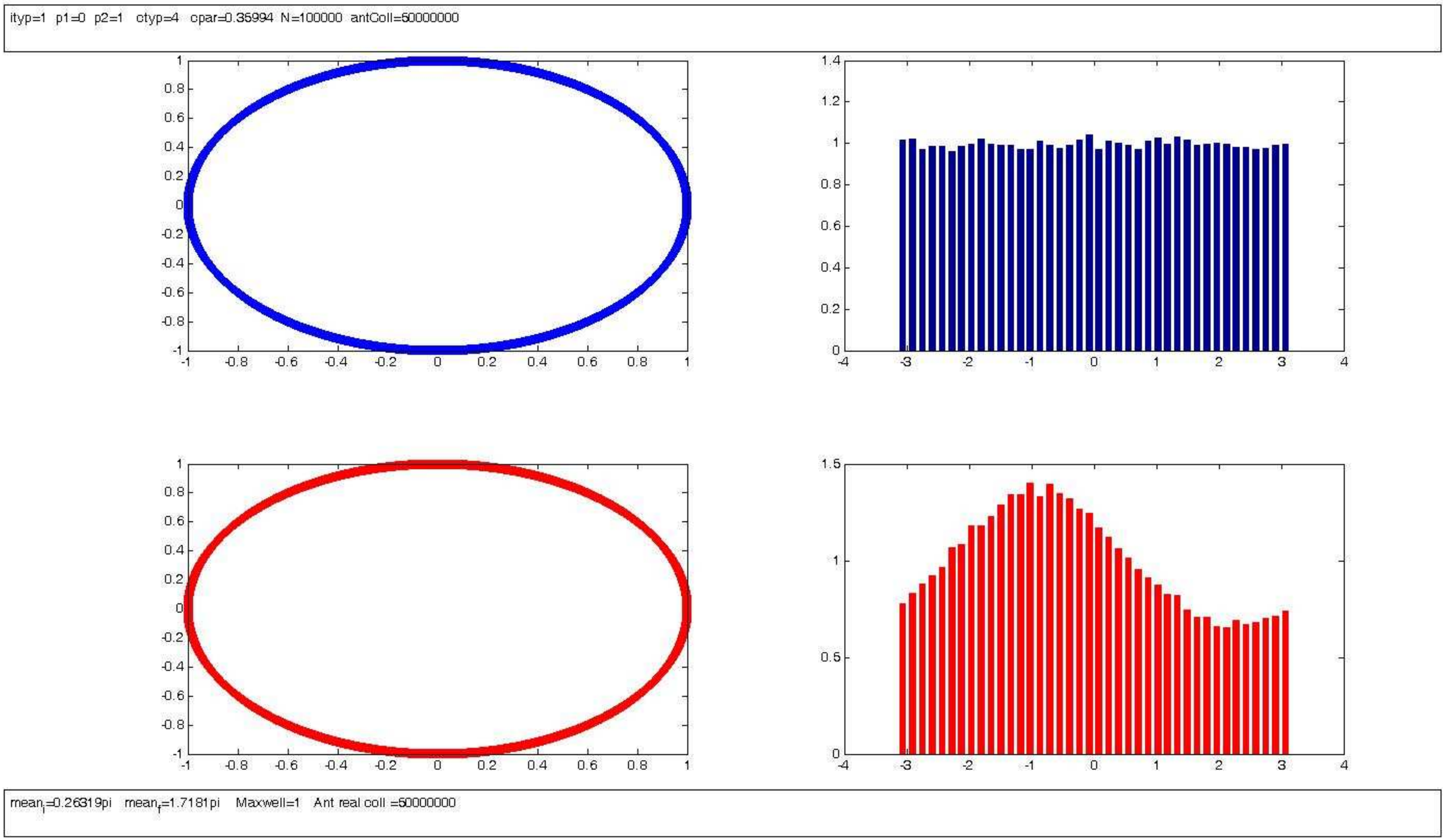
- $$\gamma_k = \frac{1}{\sum_{n=-\infty}^{\infty} \frac{a_n a_{k-n}}{a_k} \frac{\sin(\pi(n-k/2))}{\pi(n-k/2)}} \tag{1}$$
- With $a_k = e^{-\sigma^2 k^2/2}$

$$\gamma_k = e^{-\frac{\sigma^2}{4}k^2} \begin{cases} 1 & \text{when} \quad k = 2m \\ 1/(1-A) & \text{when} \quad k = 2m+1 \end{cases}, \tag{2}$$

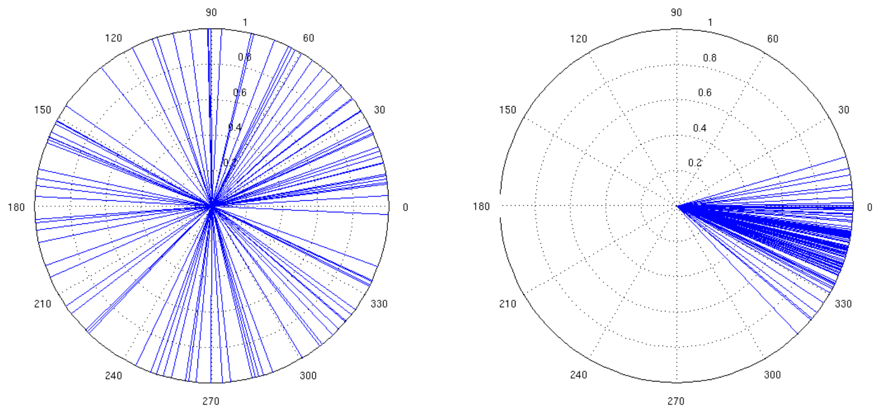
- A can be computed.

Simulation results

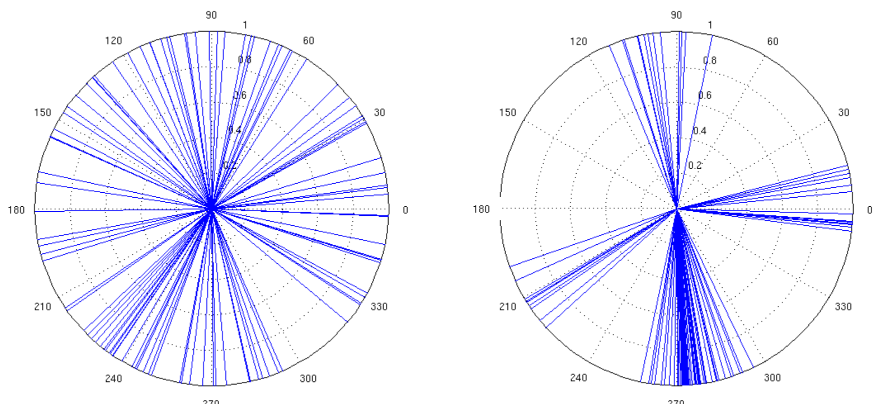


Simulation results

- Results by Robin Chatelin: Studies different models, including the BDG and CLD-models discussed here.



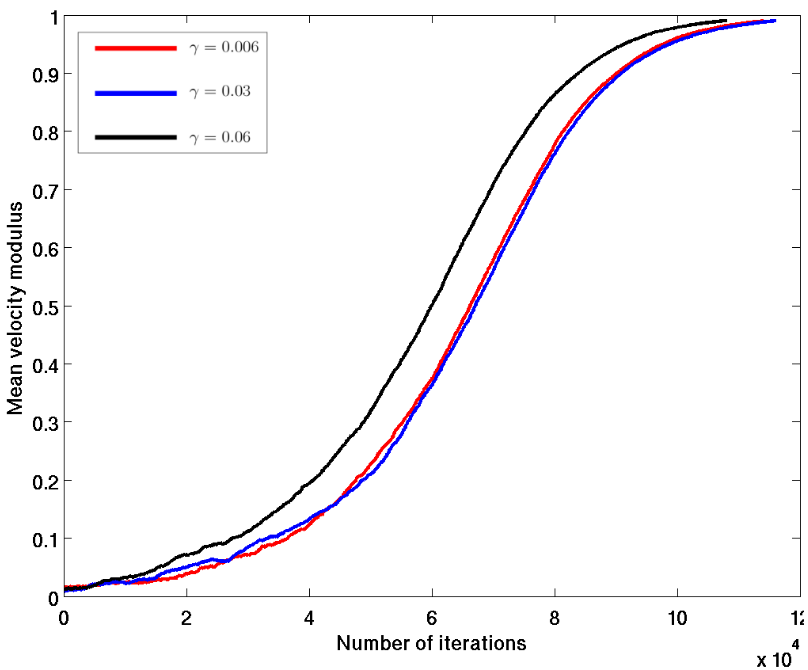
BDG-model
100 particles
t=0: uniform



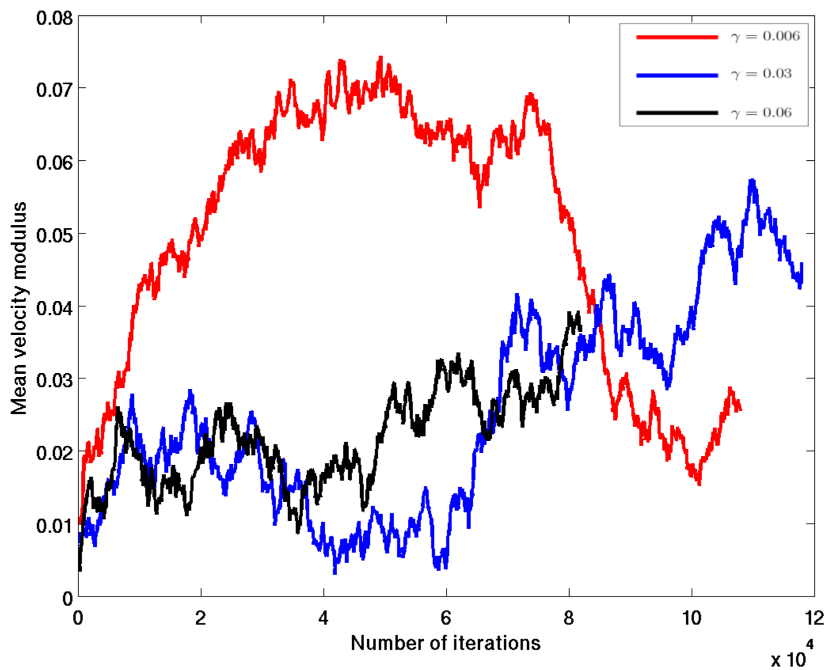
CLD-model
100 particles
t=0: uniform

Simulation results

- Mean velocity $\left|\frac{1}{N}\sum v_i\right|$.



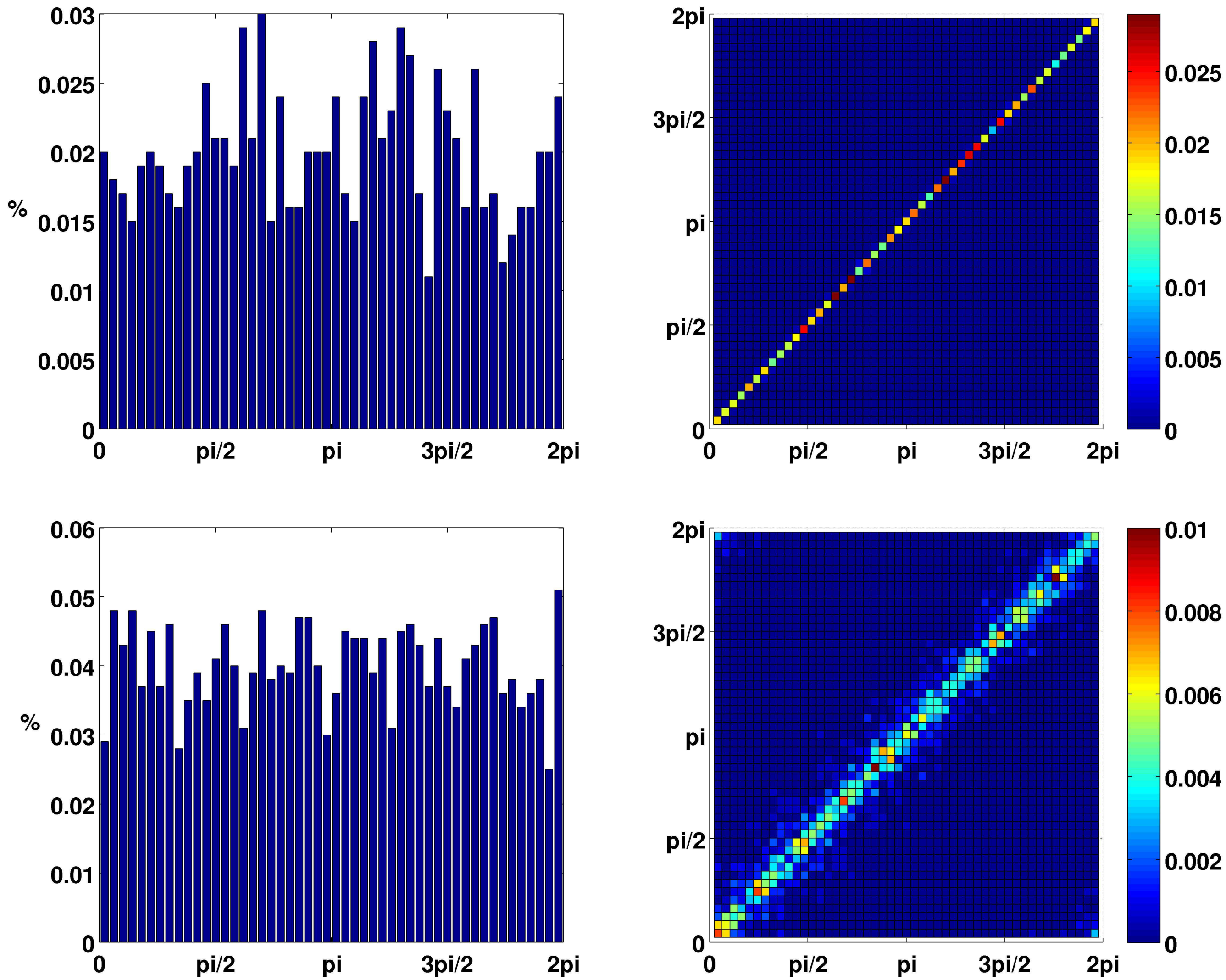
BDG model



CLD model

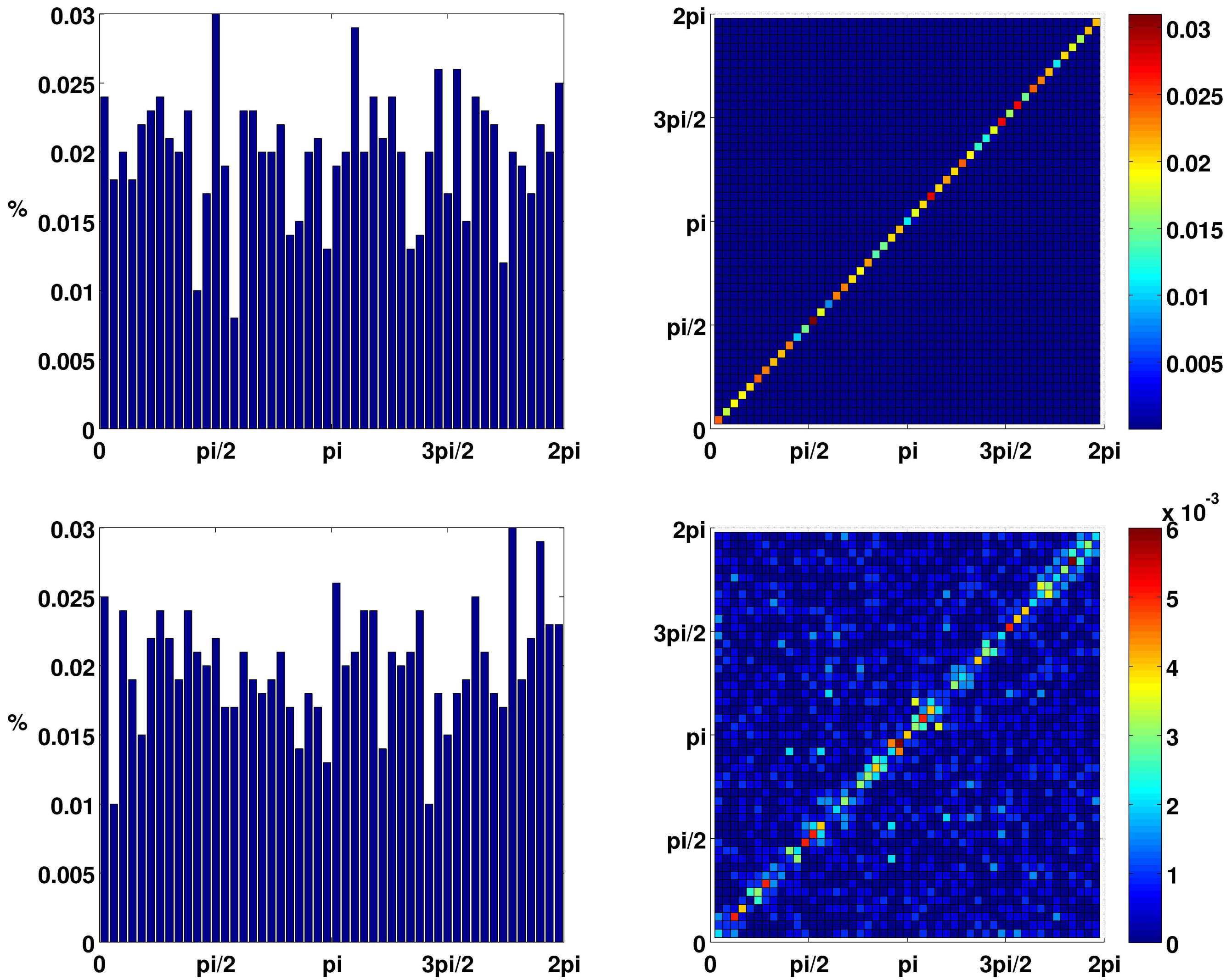
Simulation results

- Marginals. upper: BDG, lower: CLD 10^4 particles



Simulation results

- Marginals. upper: BDG, lower: CLD 10^5 particles



Thanks