# Hotspot Invasion: Traveling Wave Solutions to a Reaction-Diffusion Model for Crime Patterns

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Emergent behavior in multi-particle systems with non-local interactions
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# Introduction

- Some interesting questions:
  - Can we recreate criminal activity patterns?
  - Do innate human tendencies change the criminal activity patterns?
  - What do we expect in large time?
  - Can we observe propagation of crime?
  - Can we prevent, using minimum resources, the propagation of crime?

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- Study a basic reaction-diffusion system to model crime patterns[1]:

- $s(x,t)$ is the propensity towards crime.
- $u(x,t)$ moving average of crime.
- $c(x,t)$ cost of committing a crime.

$$\begin{aligned} s_t &= \Delta s - s + s_o(x) + (\rho(x) - c(x,t))\, u(x,t) \\ u_t &= \Lambda(s) - u(x,t) \\ c_t &= \frac{u(x,t)\rho(x)}{\int u(x,t)\rho(x)\, dx} - c(x,t) \end{aligned}$$

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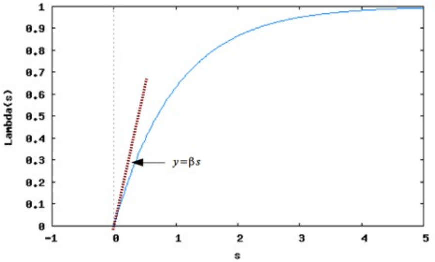
- $s_o(x)$ base willingness to commit a crime.
- $\Lambda(s)$ is the acting-out function.

$$
\Lambda(s) = \begin{cases} 0 & \text{if } s \leq 0 \\ 1 - e^{-\beta s} & \text{if } s > 0. \end{cases}
$$

- $\beta$ measures the strength that a positive $s(x,t)$ has on whether a crime is committed.



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- High-payoff policing vs. hotspot policing.

# Mathematical Formulations

- Pattern formation: Stability of steady-states (Berestycki and Nadal)
- Longtime behavior: Existence and uniqueness of steady-states.
  - Interesting case is the variable coefficient case.
  - Is there a condition which determines uniqueness of the steady-states?
- Propagation of crime: Existence of traveling wave solutions.
- Blocking invasion: Can we block propagation of crime with minimum resources?
  - What is the minimum amount of resources we need?

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  - What is the minimum amount of resources we need?

If the cost reaches a steady-state faster than the other variables, label this $c(x)$, we can define $\alpha(x) = \rho(x) - c(x)$, measures the net payoff of committing a crime.

## Exisence of Steady-States

- The model we study:

$$s_t = \Delta s - s + s_o(x) + \alpha(x)u(x,t)$$
$$u_t = \Lambda(s) - u(x,t).$$

- Solving for the steady-state solutions as $u = \Lambda(s)$ the system simplifies to

$$\boxed{\Delta s = s - s_o - \alpha(x)\Lambda(s).}$$

- For the remaining of the talk we assume that $s_o \in \mathbb{R}$ and it will provide some measure of the population tendency.

# Stability Analysis

- Let $u^*$ and $s^*$ be steady-states (not necessarily unique) and consider a perturbation

$$u(x,t) = u^* + \delta_u e^{ikx+\sigma t}$$
$$s(x,t) = s^* + \delta_s e^{ikx+\sigma t}$$

- This gives rise to the following system:

$$\begin{bmatrix} -k^2-1 & \alpha \\ \Lambda'(s^*) & -1 \end{bmatrix} \begin{bmatrix} \delta_s \\ \delta_u \end{bmatrix} = \sigma \begin{bmatrix} \delta_s \\ \delta_u \end{bmatrix}$$

- This leads to the characteristic equation

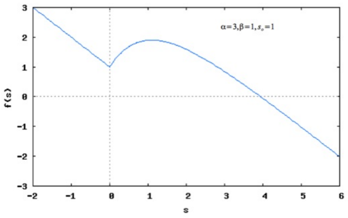
$$-\frac{(k^2+2)}{2} \pm \frac{1}{2}\sqrt{k^4 + 4\alpha\Lambda'(s^*)}.$$

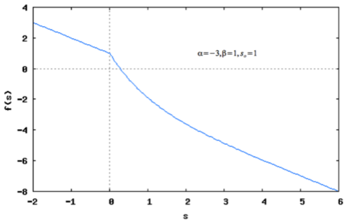
- If $\alpha\Lambda'(s^*) > 1$ this will lead to instabilities. An example is when $s_0 = 0$ and $\alpha\beta > 1$ then $s \equiv 0$ is an unstable steady-state.

# Steady-State Solutions

Consider the spatially-homogeneous case.

| $\alpha\beta$ | $s_o$ | No. of steady states |
|---|---|---|
| $< 1$ | $\mathbb{R}$ | 1 |
| $> 1$ | $> 0$ | 1 |
| $> 1$ | 0 | 2 |
| $> 1$ | $\alpha - \frac{1}{\beta}\ln(\alpha\beta) - \frac{1}{\beta}$ | 2 |
| $> 1$ | $(\alpha - \frac{1}{\beta}\ln(\alpha\beta) - \frac{1}{\beta}, 0)$ | 3 |



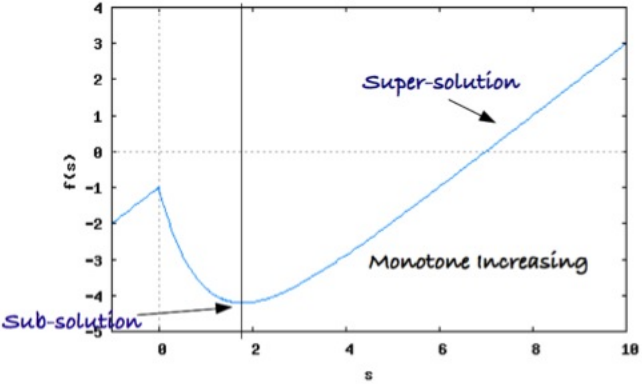




Note: $\alpha < 0$ always lead to a unique steady-state.

# $\alpha \geq 0$ and $s_0 > 0$

## Sociological Interpretation

*If there is a positive payoff for committing a crime and a natural tendency towards criminal activity, $s_0 > 0$, then one expects there to be either a hotspot or warm-spot.*



# $\alpha \geq 0$ and $s_0 = 0$

## Sociological Interpretation

*A society with a neutral tendency towards criminal activity, $s_o = 0$, will need a high enough incentive to commit a crime in order for one to observe hotspots or warm-spots.*

Critical value: $\boxed{\alpha\beta = 1}$

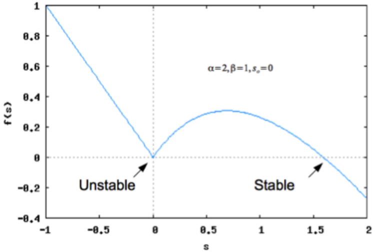



Figure: $\alpha\beta > 1$

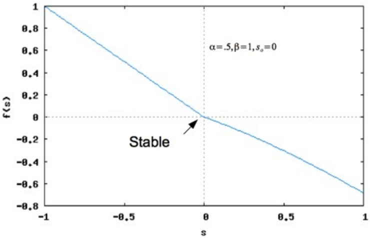



Figure: $\alpha\beta < 1$

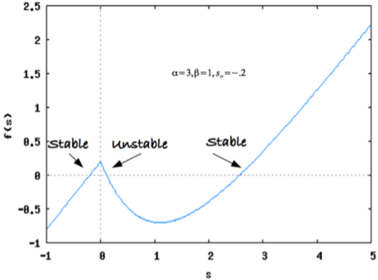
# $\alpha \geq 0$ and $s_0 < 0$

## Sociological Interpretation

*A society with a negative tendency towards criminal activities, $s_o < 0$, can exhibit interesting behavior. If the payoff to commit a crime is high enough to overcome the tendency towards peace there can be two stable steady-states and one unstable steady-state.*

- *Warm-spot* is unstable.
- Relevant condition:

$$s_o \leq \frac{1}{\beta} - \alpha + \frac{\log \alpha\beta}{\beta}.$$




Bi-stable Case

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However, the net payoff function, $\alpha(x)$, should be heterogenous.

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**Proposition (Monotone $f(u, x)$)**

*Let $\alpha(x) \leq 1/\beta$ for all $x \in \Omega$ and $s_o \in \mathbb{R}$ then there is a unique steady-state.*

- Existence:
  - $s_o \leq 0 \Rightarrow \; s_o$ is a solution.
  - $s_o > 0$: Find positive super and sub solutions.
- Uniqueness:
  - Point-wise bound leads to monotone increasing function.
  - Use Mean Value Theorem and Maximum Principle.

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How can we generalize this?

# General spatially heterogeneous payoff $\alpha(x)$

Study the spatially heterogeneous problem with $s_o = 0$:

$$\Delta s = s - \alpha(x)\Lambda(s) \quad (1)$$
$$= f(x, s) \quad (2)$$

- Clearly $s \equiv 0$ is a solution, is it unique?
- Study the eigenvalue problem:

$$\begin{cases} \Delta\phi - f_o(x)\phi = \lambda\phi \\ \phi > 0, \ \|\phi\|_\infty. \end{cases} \quad (3)$$

with $f_0(x) = \lim_{s \to 0^+} \frac{f(x,s)}{s} = 1 - \alpha(x)\beta$

## Proposition

*Let $s_0 = 0$ then $s \equiv 0$ is a solution to (1). If $\lambda > 0$, as defined in (3) then there exists a positive solution to (1). If $\lambda < 0$ then $s \equiv 0$ is the unique solution to (1).*

# Is there a sharp condition that differentiates the above cases?

- Previous condition is not very useful.
- When is $\lambda > 0$?

$$\int_\Omega f_o(x)\ dx = \int_\Omega (1 - \alpha(x)\beta)\ \ dx \le 0 \Rightarrow \lambda > 0$$

- When is $\lambda < 0$?
  - Consider

$$\Gamma \int_\Omega f_o(x)\ dx > 0,$$

    for $\Gamma > 0$
  - Then for the corresponding eigen-value problem to $\Delta s = \Gamma f(x, s)$ has a negative eigenvalue, $\lambda < 0$ for $\Gamma$ small enough.

# Ideas on the remaining cases

- $s_o > 0$ then the steady-state should be unique for general $\alpha(x)$.
- $s_o < 0$ then we have one, two, or three steady-states.
  - Natural conjecture is that the critical quantity depends on:

$$\int \left( \frac{1}{\beta} - \alpha(x) + \frac{\log \alpha(x)\beta}{\beta} - s_o \right) \, dx$$

  - Still looking for the right eigen-value problem formulations.

# Traveling Wave Solutions

**Remark**

*Toy Problem: Assume that the criminal activities reaches a steady-state much faster than the willingness to act. Is it possible for hotspots or warm-spots to invade low or zero crime regions.*

- Consider the one-dimensional problem:

$$s_t = s_{xx} - s + s_0 + \alpha\Lambda(s). \tag{4}$$

- We consider two cases:
  - $\alpha\beta \leq 1 \Rightarrow$ unique steady-state
  - $\alpha\beta > 1 \Rightarrow$ possible multiple-steady-states.

# Existence of Traveling Wave Solutions

## Sociological Interpretation

*If $\alpha\beta > 1$ one can observe the propagation of crime from high crime density ares to zero crime density areas.*

## Theorem

*Let $s_0 = 0$ and let $s(x, t)$ be a solution to . Then if*

(a) *If $\alpha\beta \leq 1$ then*

$$\lim_{t \to \infty} \|s(x, t)\|_{L^p} = 0$$

*for all $p \geq 1$.*

(b) *If $\alpha\beta > 1$ then there exists traveling wave solutions, $S(x - ct)$, connecting the two steady-states.*

# $\alpha\beta > 1$: Crime Dominates

- We see solutions of the form $S(z)$ for $c \in \mathbb{R}$ and $z = x - ct$, then

$$S'' + cS' - S + \alpha\Lambda(S) = 0,$$

such that.

$$\lim_{x\to\infty} S(x-ct) = \bar{s} > 0 \text{ and } \lim_{x\to\infty} S(x-ct) = 0.$$

- Note that we have $c\int (v_z)^2\, dz = \int_0^{\bar{s}} S - \alpha\Lambda(S)\, ds$.
- Written as a system:

$$\begin{aligned} S' &= p \\ p' &= -cp + S - \alpha\Lambda(s). \end{aligned}$$

# $\alpha\beta > 1$: Crime Dominates

- Analyze the stability of $S \equiv 0$.

$$\begin{bmatrix} 0 & 1 \\ 1-\alpha\beta & -c \end{bmatrix} \begin{bmatrix} s \\ p \end{bmatrix} = \lambda \begin{bmatrix} s \\ p \end{bmatrix}$$

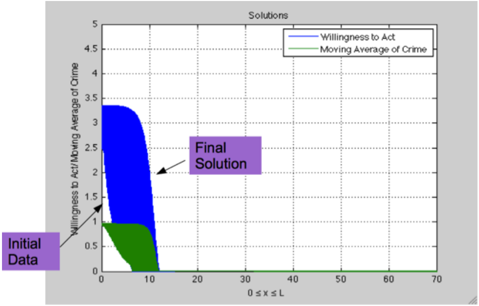
- The characteristic equation $\lambda^2 + c\lambda + \alpha\beta - 1$
- Hence, $\lambda_{\pm} = -c \pm \sqrt{c^2 - 4(\alpha\beta - 1)}$.
- Need $c \geq \sqrt{\alpha\beta - 1}$ for the eigenvalues to be real.
- The larger $\alpha\beta > 1$ the faster the the wave will travel.

# Hotspot invasion

- Consider the full one-dimensional problem:

$$s_t = s_{xx} - s + s_0 + \alpha u.$$
$$u_t = \Lambda(s) - u.$$

- Same condition on $\alpha\beta$.
- Numerical results:
  - Initial condition $\mathcal{O}(e^{-\psi x})$




$s_o = -.5, \alpha = 4$

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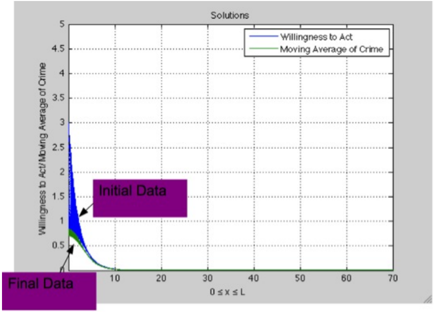


# Blocking the Invasion of Crime

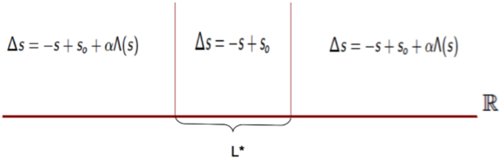
- Can we solve: $\mathbb{R}$

$$\Delta s = -s + s_o + \alpha(x)\Lambda(s)$$

with

$$\alpha(x) = \begin{cases} \alpha & |x| > L \\ 0 & |x| \leq L \end{cases}$$

- Find minimum $L$ such that the above problem has a steady state.



Thank you for your attention!