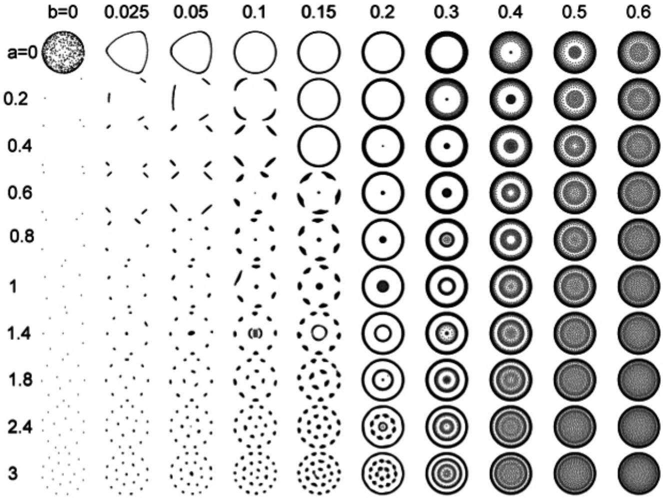
Clusters and Spots of the Aggregation Model

Mark Pavlovski

Dalhousie University

January 24, 2012



$$F(r) = min(ar + b, 1 - r)$$

Aggregation Model

$$\frac{d}{dt}x_k = \sum_{j \neq k} F(|x_k - x_j|)\frac{x_k - x_j}{|x_k - x_j|} \qquad \text{k=1...N}$$

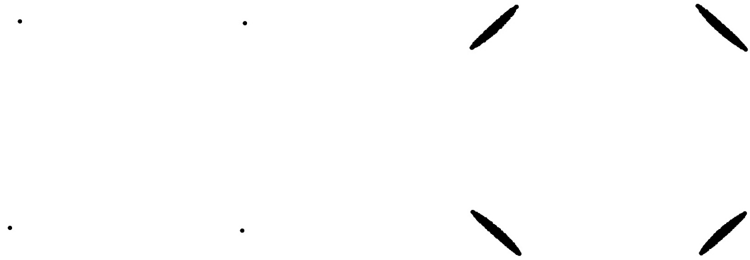
- $F(r) > 0$ for small r (repulsive)
- $F(r) < 0$ for sufficiently large r (attractive)

Simplified Model

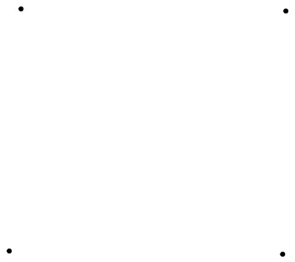
Rewriting $f(r) = F(r)/r$ yields a simpler model:

$$\frac{d}{dt}x_k = \sum_{j \neq k} f(|x_k - x_j|)(x_k - x_j) \qquad \text{k=1...N}$$

Clusters & Spots



Clusters



- Occur when $F(r) \to 0$ as $r \to 0$
- K clusters or 'holes'
- $n = \frac{N}{K}$ particles in each hole
- $x_{k,l} \in \mathbb{R}^d$, $k = 1...K$, $l = 1...n$

Clusters

We can then rewrite the system as

$$\frac{d}{dt}x_{k,l} = \sum_{g,j} f(|x_{k,l} - x_{g,j}|)(x_{k,l} - x_{h,j}) \qquad \text{k,g=1...K} \qquad \text{l,j=1...n}$$

with steady state $x_{k,l} = x_k$ satisfying

$$0 = \sum_{g} f(|x_k - x_g|)(x_k - x_g)$$

Stability

Linearize $x_{k,l} = x_k + \phi_{k,l}$, $\phi << 1$

$$\phi'_{k,l} = \sum_{g,j} [\frac{f'(|x_{k,l} - x_{g,j}|)}{|x_{k,l} - x_{g,j}|}(x_{k,l} - x_{h,j}) \otimes (x_{k,l} - x_{g,j}) + f(|x_{k,l} - x_{g,j}|)I](\phi_{k,l} - \phi_{g,j})$$

where $v \otimes u = vu^T$ and I is the identity in $\mathbb{R}^{d \times d}$

Stability

Solution decouples into two subspaces

$$\text{A:} \quad \sum_l \phi_{k,l} = 0$$

$$\text{B:} \quad \phi_{k,l} = \phi_k$$

Reduced system in A

If $\sum_l \phi_{k,l} = 0$,

$$\phi'_{k,l} = nM_k\phi_{k,l}$$

where M_k is a $d \times d$ matrix

$$M_k = \sum_g \frac{f'(|x_k - x_g|)}{|x_k - x_g|}(x_k - x_g) \otimes (x_k - x_g) + f(|x_k - x_g|)I$$

Reduced system in A in 2D

If all clusters are positioned along a ring, $x_k = re^{2\pi ik/K}$, by symmetry all K problems are identical and we write $M_k = \hat{M}$, where

$$\hat{M} = \begin{pmatrix} \hat{M}_{1,1} & 0 \\ 0 & \hat{M}_{2,2} \end{pmatrix}$$

with

$$\hat{M}_{1,1} = \sum_{g=0}^{K-1} 2rf'(2r\sin(\pi g/K))\sin(\pi g/K)\sin^2(\pi g/K) + f(2r\sin(\pi g/K))$$

$$\hat{M}_{2,2} = \sum_{g=0}^{K-1} 2rf'(2r\sin(\pi g/K))\sin(\pi g/K)\cos^2(\pi g/K) + f(2r\sin(\pi g/K))$$

Reduced system in B

If $\phi_{k,l} = \phi_k$,

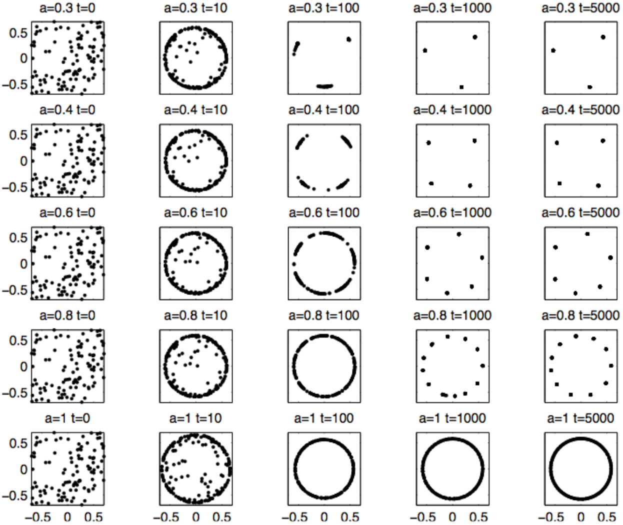
$$\phi_k' = n \sum_g \frac{[f'(|x_k - x_g|)}{|x_k - x_g|}(x_k - x_g) \otimes (x_k - x_g) + f(|x_k - x_g|)I](\phi_k - \phi_g)$$

Equivalent to stability of K holes with one particle in each hole

K-Cluster Stability Overview

The K-cluster steady state is stable iff

1. the steady state is stable when each cluster contains one particle and
2. K matrices $M_k \in \mathbb{R}^{d\times d}$ are all negative

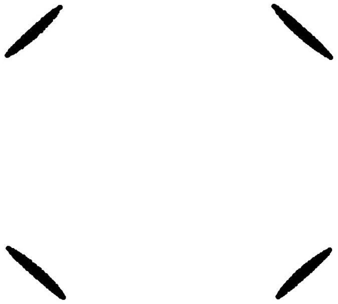

a=0.3 t=0
a=0.3 t=10
a=0.3 t=100
a=0.3 t=1000
a=0.3 t=5000
a=0.4 t=0
a=0.4 t=10
a=0.4 t=100
a=0.4 t=1000
a=0.4 t=5000
a=0.6 t=0
a=0.6 t=10
a=0.6 t=100
a=0.6 t=1000
a=0.6 t=5000
a=0.8 t=0
a=0.8 t=10
a=0.8 t=100
a=0.8 t=1000
a=0.8 t=5000
a=1 t=0
a=1 t=10
a=1 t=100
a=1 t=1000
a=1 t=5000
0.5
0
–0.5
–0.5 0 0.5

$$f(r) = \min(a, 1 - r)$$

The K-cluster solution is guaranteed to be stable if $a \in (a1, a2)$

K	r	a_1	a_2
3	$3^{-1/2} \approx 0.5773$	0	0.5
4	0.585786	0.171573	0.656851
5	0.587785	0.309017	0.736067
6	0.588457	0.411543	0.788636
7	0.588735	0.489115	0.819194
8	0.588867	0.549301	0.841735
$\gg 1$	$\frac{3}{16}\pi$	$1 - \frac{3\pi^2}{8K}$	$1 - \frac{\pi^2}{8K}$

Spots



- Occur when $F(r) > 0$ at the origin
- K spots with $n = \frac{N}{K}$ particles in each spot
- $F(r) = \delta$ at the origin, $0 < \delta \ll 1$
- $F'(0) = a, a > 0$

Spots Steady State

Constructing spots solution as a perturbation of the clusters solution:

$$x_{k,l} = x_k + \delta\phi_{k,l}$$

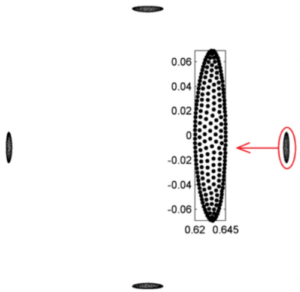
and considering self-consistent anzatz $\sum_l \phi_{k,l} = 0$ yields the following reduced system

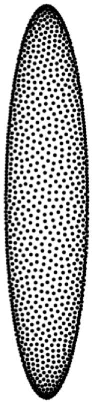
$$\phi_{k,l} = \sum_{j \neq l} \frac{\phi_l - \phi_j}{|\phi_l - \phi_j|} + nM_k\phi_{k,l}$$

with

$$M_k = aI + \sum_g \frac{f'(|x_k - x_g|)}{|x_k - x_g|}(x_k - x_g) \otimes (x_k - x_g) + f(|x_k - x_g|)I$$

Reduced system in 2D





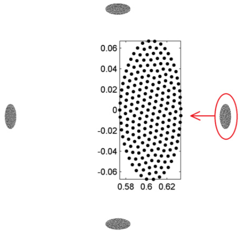
- K spots of equal size symmetrically distributed along a ring
- fix arbitrary $k = K$, $x_K = (r, 0)$
- $\phi_{K,l} = \phi_l$

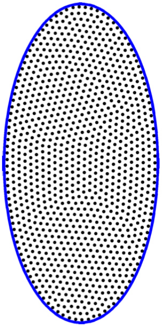
$$\phi_l = \sum_{j \neq l} \frac{\phi_l - \phi_j}{|\phi_l - \phi_j|} - n \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \phi_l$$

$$\alpha = -a - \sum_{g=1}^{K-1} 2rf'(2r\sin(\pi g/K))\sin(\pi g/K)\sin^2(\pi g/K) + f(2r\sin(\pi g/K))$$

$$\beta = -a - \sum_{g=1}^{K-1} 2rf'(2r\sin(\pi g/K))\sin(\pi g/K)\cos^2(\pi g/K) + f(2r\sin(\pi g/K))$$

Spots of Uniform Density





$$f(r) = a + \frac{\delta}{r^2}$$

Continuous model:

$$\rho_t(x,t) + \nabla_x \cdot (v(x)\rho(x,t)) = 0$$

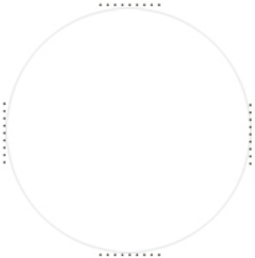
$$v(x) = \int_{\mathbb{R}^2} \{\nabla_x \ln|x-y| - A\frac{\nabla_x|x-y|^2}{2}\}\rho(y)dy, A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

Let D be an ellipse, $\partial D = (acos\theta, bsin\theta)$, $\theta = 0...2\pi$

$$a = \sqrt{\frac{2}{\alpha+\beta}\frac{\beta}{\alpha}}, b = \sqrt{\frac{2}{\alpha+\beta}\frac{\alpha}{\beta}}$$

The system admits a steady state ρ=constant inside D and 0 outside D

Stripe Steady State



The system admits a '1 dimensional' stripe solution'

$$\phi_j = (\frac{2}{\beta n} j - \frac{1}{\beta}(1 + 1/n))i$$

Stability of a stripe

Perturb

$$\phi_j = (\frac{2}{\beta n} j - \frac{1}{\beta}(1 + 1/n))i + \psi_j \qquad \psi_j << 1$$

Linearized system becomes

$$\psi_l' = \sum_{j \neq l} \frac{\beta n}{4|l-j|}(\bar{\psi}_l - \bar{\psi}_j + \psi_l - \psi_j) - n(\frac{\alpha+\beta}{2}\phi_l + \frac{\alpha-\beta}{2}\bar{\psi}_l)$$

Two solution subspaces

Vertical Perturbation

$$\Re(\psi_I) = 0$$

Horizontal Perturbation

$$\Im(\psi_I) = 0$$

Vertical Perturbation

Reduces to a 1-D problem. Let $\psi_I = e^{\lambda t} i$. Then we have

$$\lambda = -n\beta$$

Horizontal Perturbation

Reduces to

$$\psi_l' = \sum_{j \neq l} \frac{\beta n}{2|l-j|}(\psi_l - \psi_j) - n\alpha\psi_j$$

Key Lemma

The n eigenvalues for the problem

$$\lambda\psi_l = \sum_{j\neq l} \frac{\psi_l - \psi_j}{|l-j|} \qquad l = 1 \ldots n$$

are given by

$$\lambda_k = 2\sum_{j=1}^{k} \frac{1}{j}, \qquad k = 0 \ldots n-1$$

$$\sum_{j=1}^{n} \frac{1}{j} \to ln(n) + \gamma \text{ as } n \to \infty \qquad (\gamma = 0.5772156\ )$$

The stripe is stable if

$$\frac{\alpha}{\beta} > S_{n-1}$$

Or as $n \to \infty$

$$n < e^{\frac{\alpha}{\beta}-\gamma} \approx 0.5614 e^{\frac{\alpha}{\beta}}$$