Models of flocking with asymmetric interactions

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joint work with

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Emergent behavior in multi-particle systems
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Outline

What is flocking?

Nature gives many examples of flocking behavior.

There are two characteristics in a flock:

- the distance between individuals remains bounded (**bounded distance**),
- they all move in the same direction (**alignment**).

Outline

1. A model with asymmetric interactions
   - The Cucker-Smale model
   - Drawbacks of the C-S model
   - A model with asymmetric interactions
2. Flocking for the new model
   - ℓ^∞ approach
   - Condition for flocking
   - Extension

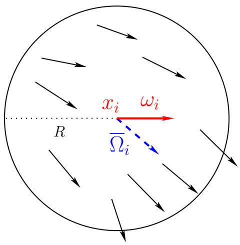
The Vicsek model

Discrete Vicsek model ('95)

$$x_i^{n+1} = x_i^n + \Delta t \, \omega_i^n$$

$$\omega_i^{n+1} = \overline{\Omega}_i^n + \epsilon$$

with $\overline{\Omega}_i^n = \dfrac{\sum_{|x_j - x_i| < R} \omega_j^n}{\left|\sum_{|x_j - x_i| < R} \omega_j^n\right|}$, ϵ noise.



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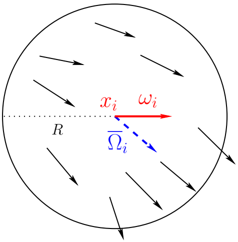

Continuous Vicsek model ('08 Degond, M.)

$$\frac{dx_i}{dt} = \omega_i$$

$$d\omega_i = (\mathrm{Id} - \omega_i \otimes \omega_i)(\nu\, \overline{\Omega}_i\, dt + \sqrt{2D}\, dB_t)$$

Video

The Cucker-Smale model

Cucker and Smale proposed a simple model:

- no noise ($\epsilon = 0$),
- no constraint on the velocity ($|\omega_i| \neq 1$),
- the mean velocity ($\overline{\Omega}_i$) is simply a sum.

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Cucker-Smale model '07

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \qquad \frac{d\mathbf{v}_i}{dt} = \frac{\alpha}{N}\sum_{j=1}^{N} \phi_{ij}(\mathbf{v}_j - \mathbf{v}_i), \tag{1}$$

where $\alpha > 0$ and ϕ_{ij} is the *influence* of agent j on agent i:

$$\phi_{ij} := \phi(|\mathbf{x}_j - \mathbf{x}_i|).$$

with $\phi(\cdot)$ a positive decreasing function (ex: $\phi(r) = \frac{1}{1+r}$).

Video

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Flocking for the C-S model

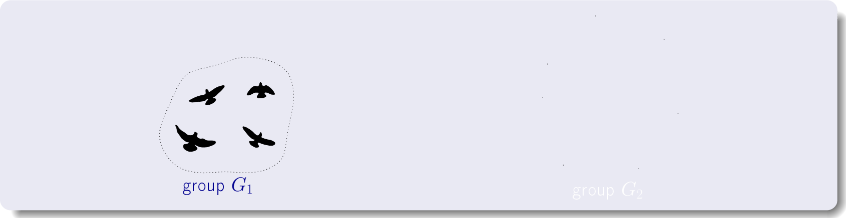
Thm. *If the influence function ϕ decays slowly enough:*

$$\int_0^\infty \phi(r)\, dr = +\infty,$$

then the C-S model converges to a flock.

Ref. Cucker-Smale ('07), Ha-Tadmor ('08), Carrillo-Fornasier-Rosado-Toscani ('09), Ha-Liu ('09).

Drawbacks of the C-S model



In the "small" group G_1 alone:

$$\frac{d\mathbf{v}_i}{dt} = \frac{\alpha}{N_1} \sum_{j=1}^{N_1} \phi_{ij} (\mathbf{v}_j - \mathbf{v}_i)$$

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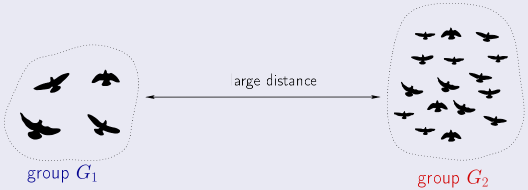

In the "small" group G_1 with the "large" group G_2:

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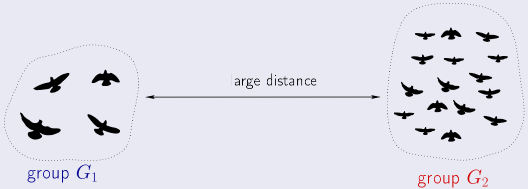

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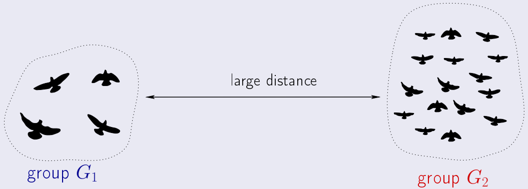

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We propose the following dynamical system:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \qquad \frac{d\mathbf{v}_i}{dt} = \frac{\alpha}{\sum_{k=1}^{N} \phi_{ik}} \sum_{j=1}^{N} \phi_{ij} \left(\mathbf{v}_j - \mathbf{v}_i\right), \qquad (2)$$

with $\phi_{ij} = \phi(|\mathbf{x}_j - \mathbf{x}_i|)$ and $\alpha > 0$.

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The influence of the agent j on agent i is weighted by **the total influence**, $\sum_{k=1}^{N} \phi_{ik}$, exerted on agent i.

Remark. If $\phi_{ij} \approx \phi_0 \Rightarrow$ the C-S dynamics. Otherwise the model better captures strongly "*non-homogeneous*" scenario.

Asymmetric interactions

The model can be written as:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \qquad \frac{d\mathbf{v}_i}{dt} = \alpha \sum_{j=1}^{N} a_{ij}(\mathbf{v}_j - \mathbf{v}_i),$$

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with:

$$a_{ij} := \frac{\phi(|\mathbf{x}_j - \mathbf{x}_i|)}{\sum_{k=1}^{N} \phi(|\mathbf{x}_k - \mathbf{x}_i|)} \geq 0 \quad , \qquad \sum_j a_{ij} = 1.$$

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The total momentum ($\overline{\mathbf{v}} = \frac{1}{N}\sum_i \mathbf{v}_i$) is **not preserved** in the model!

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ℓ^∞ approach

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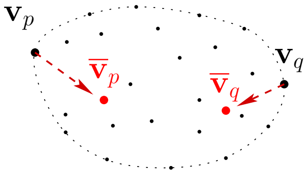
Take p, q such that:

$$d_V = |\mathbf{v}_p - \mathbf{v}_q|$$



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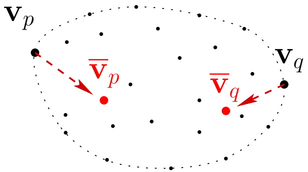

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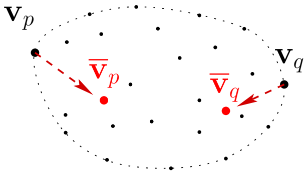

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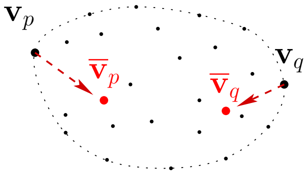

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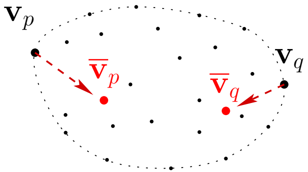

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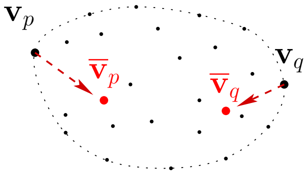

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Lemma

Lemma. Let S be an *antisymmetric* matrix bounded by M, u, w be two positive vectors ($u_i, w_i \geq 0$) satisfying $\sum_i u_i = \sum_j w_j = 1$. Then,

$$|\sum_{i,j} S_{ij} u_i w_j| \leq M$$

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$$|\sum_{i,j} S_{ij} u_i w_j| \leq M \left(1 - \lambda^2(\theta)\, \theta^2\right),$$

where $\lambda(\theta)$ denotes the number of **"active entries"**

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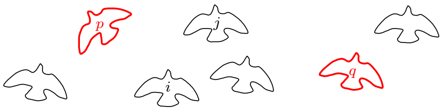

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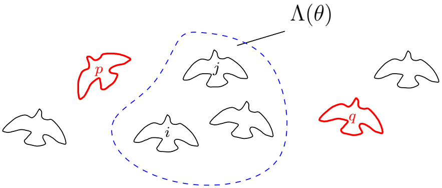

Proposition

Fix $\theta > 0$. Then the diameters $d_X(t)$ and $d_V(t)$ satisfy,

$$\frac{d}{dt} d_X(t) \leq d_V(t) \quad , \quad \frac{d}{dt} d_V(t) \leq -\alpha\, \lambda^2(\theta)\, \theta^2\, d_V(t).$$

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To conclude we need to find an appropriate θ for which we can *count* the number of "active entries" $\lambda(\theta)$.

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$\Rightarrow$ $d_V(t) \to 0$ expo. fast.

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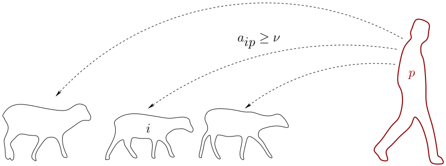

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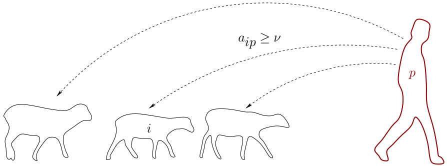

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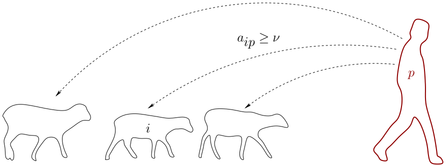

  - Cucker-Smale model
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- Kinetic and macroscopic equations
  - Kinetic equation: $\partial_t f + v \cdot \nabla_x f + \nabla_v (Ff) = 0$.
  - Fluid equation:

$$\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0$$
$$\partial_t (\rho \mathbf{u}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \mathbf{S}(\mathbf{u}).$$

Conclusion & Perspectives

Summary

- Introduction of a asymmetric model of flocking

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Perspectives

- Existence and uniqueness for the kinetic equation

  joint work with E. Boissard

- Study the dynamics when ϕ has only a compact support

  joint work with E. Tadmor