Stationary states for the aggregation equation with power law attractive-repulsive potentials

Daniel Balagué

Joint work with J.A. Carrillo, T. Laurent and G. Raoul

Universitat Autònoma de Barcelona

BIRS - Multi-particle systems with non-local interactions
22-27 January 2012 - Banff, Canada

We consider the problem given by

$$\rho_t = -\mathrm{div}[\rho v] \quad \text{in} \quad \mathbb{R}^N \times [0, T]$$
$$v = -\nabla W * \rho$$
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Introduction

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Here $\rho(x, t)$ is a density of particles located at position x at time t and W is a given interaction potential.

What do we know about the solutions?

- Existence and uniqueness of weak solutions for the aggregation equation in $\mathcal{P}_2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, for $\rho_0 \in L^p$ and $\nabla W \in W^{1,q}$.

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- Global existence when ΔW is bounded from above.
- Weak measure solutions to the the Cauchy problem for the aggregation equation.
- The problem is global-in-time well-posed in $L^p(\mathbb{R}^N)$ under Osgood conditions, and there is blow-up of the L^p-norm when this condition is violated.

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Definition (Spherical shell)

A delta on a sphere of radius R ("spherical shell", $\delta_{\partial B(0,R)}$), denoted it by δ_R, is a uniform distribution on a sphere $\partial B(0, R) = \{x \in \mathbb{R}^N \,:\, |x| = R\}$.

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Remark

Then μ can be written as a sum of δ_R, $\int_0^\infty \delta_r d\hat{\mu}(r)$. And,

$$v = -(\nabla W * \mu)(x) = \int_0^\infty \omega(|x|, r) d\hat{\mu}(r) \frac{x}{|x|}.$$

The problem in radial coordinates

In radially symmetric coordinates, the equation reads:

$$\partial_t \hat{\mu} + \partial_r(\hat{\mu}\hat{v}) = 0,$$
$$\hat{v}(r,t) = \int_0^{+\infty} \omega(r,\eta)\, d\hat{\mu}_t(\eta).$$

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The function ω is defined by

$$\omega(r,\eta) = -\frac{1}{\sigma_N} \int_{\partial B(0,1)} \nabla W(re_1 - \eta y) \cdot e_1 d\sigma(y),$$

where σ_N is the surface area of the unit ball in $\mathbb{R}^N$ and e_1 is the first vector of the canonical basis of $\mathbb{R}^N$.

Stationary states

Definition

A probability measure $\mu \in \mathcal{P}(\mathbb{R}^N)$ is a stationary state for the aggregation equation if

$$-(\nabla W * \mu)(x) = 0 \quad \text{for all} \quad x \in \mathrm{supp}(\mu).$$

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According to the given interpretation of the velocity field, a δ_R is a stationary state for the aggregation equation if and only if

$$\omega(R, R) = 0.$$

Energetic point of view

The aggregation equation is a gradient flow[1] of the following energy functional

$$E[\mu] = \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y) d\mu(x) d\mu(y)$$

w.r.t. the euclidean Wasserstein distance

$$d_2^2(\nu, \rho) = \inf_{\pi \in \Pi(\nu,\rho)} \left\{ \iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^2 d\pi(x,y) \right\},$$

where $\Pi(\nu, \rho)$ stands for the set of joint distributions with marginals ν i ρ.

[1] Ambrosio, L. A.; Gigli, N.; Savaré, G. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics, Birkhäuser, 2005.

Then, the stationary states asymptotically stable are local minimizers of the energy. Let us see the conditions for a δ_R

Proposition

Suppose $\omega \in C^1$ and let δ_R be a stationary state, $\omega(R, R) = 0$. then,

(i) If $\partial_1 \omega(R, R) > 0$ exists $dr_0 > 0$ such that, given $0 < |dr| < dr_0$,

$$E[(1 - \epsilon)\delta_R + \epsilon \delta_{R+dr}] < E[\delta_R],$$

for ϵ small enough.

(ii) If $\partial_1 \omega(R, R) + \partial_2(R, R) > 0$ exists $dr_0 > 0$ such that

$$E[\delta_{R+dr_0}] < E[\delta_R],$$

for all $0 < |dr| < dr_0$.

Instability conditions

Theorem

Suppose that δ_R is a stationary state, $\omega(R,R)=0$, and that one of the following cases is satisfied:

(i) $\omega \in C^1(\mathbb{R}^2_+)$ and $\partial_1\omega(R,R) > 0$.

(ii) $\omega \in C(\mathbb{R}^2_+) \cap C^1(\mathbb{R}^2_+ \setminus \mathcal{D})$ and

$$\lim_{\substack{(r_1,r_2)\notin\mathcal{D}\\(r_1,r_2)\to(R,R)}} \partial_1\omega(r_1,r_2) = +\infty.$$

Conclusion: It is not possible for an L^p radially symmetric solution to converge toward a δ_R when $t \to \infty$.

Idea for the instability

One first observes that

$$(\mathrm{div} v)(x) = -\Delta W * \delta_R(x) = \partial_1 \omega(|x|, R) + (N-1)\frac{\omega(|x|, R)}{r_1}.$$

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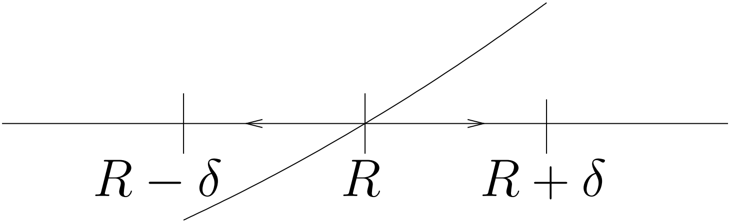

Conditions for the stability

For the stability we suppose that $\omega \in C^1$ and that δ_R is an stationary state. Moreover, suppose that

(i) $\partial_1 \omega(R, R) < 0$,

(ii) $\partial_1 \omega(R, R) + \partial_2 \omega(R, R) < 0$.

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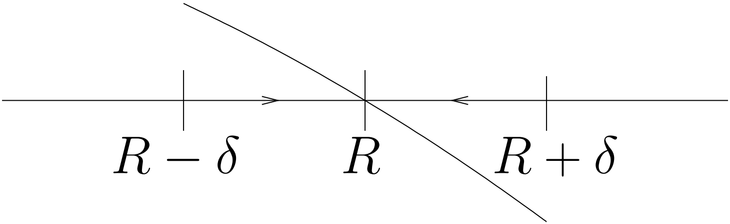
The interpretation is similar...



Instability: non-radial case

Consider an steady state $\overline{\mu}$ of the form

$$\int_{\mathbb{R}^N} f(x)\, d\overline{\mu}(x) = \int_{\mathcal{M}} f(x)\phi(x)\, d\sigma(x), \quad \forall f \in C(\mathbb{R}^N),$$

where $\mathcal{M}$ is a C^2 hypersurface and $d\sigma$ is the volume element on $\mathcal{M}$.

Proposition

Suppose that one of the two conditions holds:

(i) ΔW is locally integrable on hypersurfaces, $\phi \in L^\infty(\mathcal{M})$ and

$$(\operatorname{div} \overline{v}(x) := -(\Delta W * \overline{\mu})(x) > 0 \quad \forall x \in \operatorname{supp}(\overline{\mu}),$$

(ii) ΔW is not locally integrable on hypersurfaces and $\phi(x) > \phi_0 > 0$ for all $x \in \mathcal{M}$.

Conclusion: it is not possible for an L^p solution to converge to $\overline{\mu}$ w.r.t. the d_∞-topology.

An example with powers

Consider now power law potentials:

$$W(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b} \qquad 2 - N < b < a$$

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Some properties:

- The condition $b < a$ ensures that the potential is repulsive in the short range and attractive in the long range.
- The condition $2 - N < b$ ensures that the potential is in $W^{1,q}_{loc}(\mathbb{R}^N)$ for some $1 < q < \infty$.

δ_R in the power law case

For the case of powers one has a formula for the function $\omega(r,\eta)$:

$$\omega(r,\eta) = r^{b-1}\psi_b(\eta/r) - r^{a-1}\psi_a(\eta/r),$$

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where

$$\psi_a(s) = \frac{\sigma_{N-1}}{\sigma_N}\int_0^{\pi} \frac{(1 - s\cos\theta)(\sin\theta)^{N-2}}{(1 + s^2 - 2s\cos\theta)^{\frac{2-a}{2}}} d\theta.$$

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Then, a δ_R is a stationary state for the aggregation equation if and only if

$$\omega(R, R) = 0, \quad R = R_{ab} = \left(\frac{\psi_b(1)}{\psi_a(1)}\right)^{\frac{1}{a-b}}.$$

Value of R?

In the power law case, the radius R can be computed explicitly:

$$R = R_{ab} = \frac{1}{2}\left(\frac{\beta(\frac{b+N-1}{2}, \frac{N-1}{2})}{\beta(\frac{a+N-1}{2}, \frac{N-1}{2})}\right)^{\frac{1}{a-b}}$$

In/stability for powers

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(i) If $2 - N < b \leq 3 - N$ then $\omega \in C(\mathbb{R}^2_+) \cap C^1(\mathbb{R}^2_+ \setminus \mathcal{D})$ and for all $(R, R) \in \mathcal{D}$

$$\lim_{\substack{(r_1,r_2)\notin\mathcal{D} \\ (r_1,r_2)\to(R,R)}} \partial_1\omega(r_1, r_2) = +\infty.$$

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(ii) If $b \in \left(3 - N, \frac{(3-N)a-10+7N-N^2}{a+N-3}\right)$ then $\omega \in C^1$ and

$$\partial_1\omega(R_{ab}, R_{ab}) > 0.$$

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(iii) If $b \in \left(\frac{(3-N)a-10+7N-N^2}{a+N-3}, a\right)$ then $\omega \in C^1$ and

$$\partial_1\omega(R_{ab}, R_{ab}) < 0 \quad \text{i} \quad \partial_1\omega(R_{ab}, R_{ab}) + \partial_2\omega(R_{ab}, R_{ab}) < 0.$$

Idea of the proof (i)

(i) **Case** $2 - N < b \leq 3 - N$. One has $\omega \in C^1((\mathbb{R}^+ \times \mathbb{R}^+)\backslash\mathcal{D})$ and

$$\frac{\partial \omega}{\partial r}(r,\eta) = r^{b-2}\Big[(b-1)\psi_b(\eta/r) - (\eta/r)\psi_b'(\eta/r)\Big]$$
$$-r^{a-2}\Big[(a-1)\psi_a(\eta/r) - (\eta/r)\psi_a'(\eta/r)\Big] \qquad \text{in } (\mathbb{R}^+,\mathbb{R}^+)\backslash\mathcal{D}.$$

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Calculating the limit,

$$\lim_{\substack{(r,\eta)\to(R,R)\\(r,\eta)\notin\mathcal{D}}}\frac{\partial\omega}{\partial r}(r,\eta)=+\infty.$$

Idea of the proof (ii)

(ii) **Case** $3 - N < b < \frac{(3-N)a-10+7N-N^2}{a+N-3}$. In this case, $\omega \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$.

Idea of the proof (ii)

(ii) **Case** $3 - N < b < \frac{(3-N)a-10+7N-N^2}{a+N-3}$. In this case, $\omega \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$. Condition

$$\frac{\partial \omega}{\partial r}(R_{ab}, R_{ab}) > 0 \Longleftrightarrow a - \frac{\psi_a'(1)}{\psi_a(1)} < b - \frac{\psi_b'(1)}{\psi_b(1)}$$

Idea of the proof (iii)

A change of variable in the expression of $\psi_a(1)$ shows that

$$\psi_a(1) = \frac{\omega_{N-1}}{\omega_N} 2^{a+N-3} \beta\left(\frac{a+N-1}{2}, \frac{N-1}{2}\right),$$

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and for $\psi_a'(1)$

$$\psi_a'(1) = \frac{\omega_{N-1}}{\omega_N} \frac{(a-2)(a+N-2)}{N-1} 2^{N+a-4} \beta \left(\frac{a+N-3}{2}, \frac{N+1}{2} \right).$$

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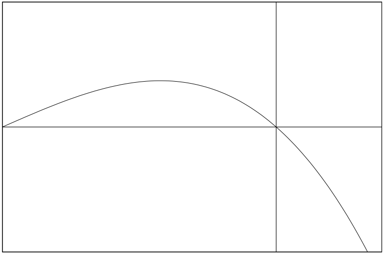
Then the quotient $\frac{\psi_a'(1)}{\psi_a(1)}$ becomes

$$\frac{\psi_a'(1)}{\psi_a(1)} = \frac{1}{2} \frac{(a-2)(a+N-2)}{a+N-3}.$$

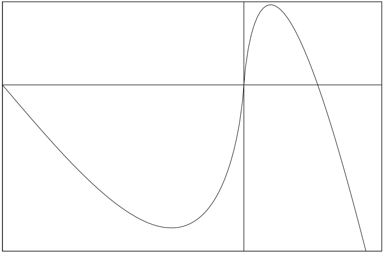
□

Some pictures for the velocity field $\omega(r, R_{ab}), N = 2$

a=4, b=2, Rab=0.5773502691896258



a=2, b=1, Rab=0.6366197723675802



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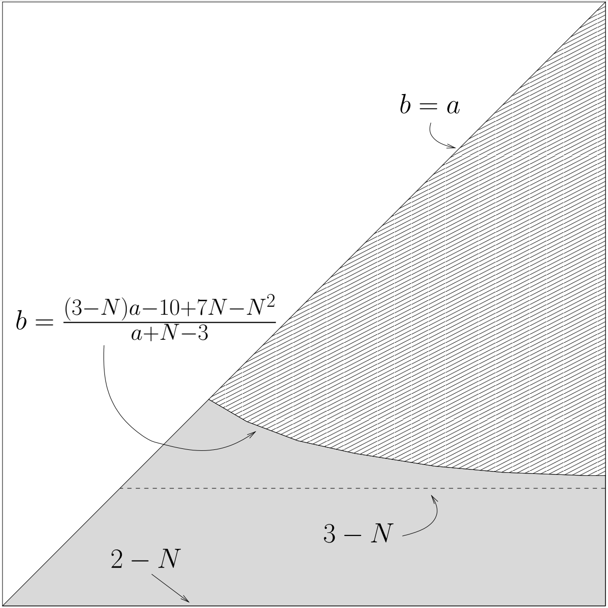

Thank you for your attention.