A refined flocking and swarming model of Cucker-Smale type

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Outline

- Present a more realistic (?) particle system, based on the *Cucker-Smale* alignment model, that describes the collective behavior of swarms (e.g., birds, fish, insects, mammals).

  Here, we'll talk about birds. But our model adapts to other types of swarms (e.g., fish, mammals, ...); the interaction rules will need adjustments from case to case.

- Derive the associated kinetic equation, useful when the number of birds is very large.

- Discuss well-posedness, and the convergence of the particle model to the kinetic equation as the number of birds grows very large.

- Numerical simulations in 3-D of our particle model.

Basic rules in models for swarming

Most of the basic particle models for swarming are based on three behavioral rules between animals: rules:

- **short-range repulsion.** Birds in the swarm keep a minimum distance between each other so as to move freely; this explains why collisions are avoided.

- **Alignment or Orientation.** Birds tend to align themselves with those at intermediate distances; this explains how they produce synchronized structures.

- **Long-range attraction.** Birds are attracted to those at farther distances. This explains why they move in groups.

These basic rules are used as building blocks to make up other more realistic particle models, by simply including in the basic models some more realistic features.

Basic particle models

- There are many basic swarming particle models in the biology literature; e.g., *the three-zone models* by [Aoki, 1982], [Huth & Wissel, 1992], [Couzin et al., 2002], ...

- Another basic model often used in the mathematics literature is the *Cucker-Smale model*. It is similar to existing models of interacting particles, and leads to a Vlasov-like kinetic equation.

- **Cucker-Smale model (2007).** It is an alignment model where the velocity of each bird is updated by taking a weighted average of its relative velocities w.r.t. all the birds in the flock; and the weight of this averaging process takes into account the distances between the birds, in a way that closer birds have stronger influence than far distant ones.

The Cucker-Smale model

- **Particle model.** The position $x_i(t)$ and velocity $v_i(t)$ of a bird i vary according to:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} H\left(|x_i - x_j|\right)(v_j - v_i) \end{cases}$$

with $H(r) = \frac{\gamma}{(1+r^2)^\sigma}$ for some parameters $\gamma, \sigma > 0$.

- This system is the discrete version of the **kinetic eqn.**

$$\partial_t f + v \cdot \nabla_x f = -\nabla_v \cdot [\xi(f)f], \qquad \text{where}$$

$$\xi(f)(t, x, v) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} H\left(|x - y|\right)(w - v)f(t, y, w)dydw.$$

- Both equations are the same; the former is a special case of the latter, arising in the case where f is a discrete measure given by $\sum \delta(x - x_i(t))\delta(v - v_i(t))$.

Some known results about this model

- **Momentum conservation.** (Cucker, Smale, Ha, Liu, Tadmor, ...): $\frac{d}{dt}\left(\sum_{i=1}^{N} v_i(t)\right) = 0$.

- **Exponential convergence of the flock for $\sigma \le 1/2$.** (Cucker, Smale, Ha, Liu, Carrillo, Fornasier, Rosado, Toscani, ...): *When $t \to \infty$ then $v_i(t) \to \bar{v}$ exponentially fast for all i, where $\bar{v}$ is the initial mean velocity of all the birds. Moreover, $|x_i(t) - (\bar{x} + \bar{v}t)| < \epsilon$ for all i, where $\epsilon > 0$ depends only on the initial configuration of the swarm, and $\bar{x}$ is the initial center of mass of the swarm.* In other words, in the long time, all the birds will move in a single group and with the same velocity (*this is called "unconditional flocking"*).

- The key ingredient in the proofs is the *symmetry* of the interaction rule in the Cucker-Smale model, namely, *the force that bird i exerts on bird j is the same as that of bird j onto i.*

Weakness of this model

- While the "*unconditional flocking*" is a nice feature to emerge from a model, it is NOT what we generally see in swarms.
  Observations suggest that while a swarm will persist for a long time, NOT all birds will in general acquire the same velocity, and NOT all of them will in general move in one group. Swarms undulate, grow branches, disperse, merge, and so on. So, in general, splitting of a swarm into many groups is possible.

- Therefore, *the symmetry* of the interaction rule in the Cucker-Smale model is an excessively strong constraint. *Here is an example of 2 birds*: bird 1 following bird 2, but bird 2 does not see bird 1 and therefore does not react to its presence. Hence their interactions are not symmetric, and momentum is not conserved.

Refinements

Our modified Cucker-Smale model are based on 4 specific local rules; *it does NOT conserve momentum*, and therefore "unconditional flocking" does NOT hold. These rules are:

- **Friction.** There is a cruise speed that every bird attempts to fly with (in the absence of other forces). It is denoted by $\sqrt{\alpha/\beta}$ where $\alpha, \beta > 0$ are some parameters.

- **Repulsion.** Birds keep a minimum distance (*denoted by d_0*) between each other at all times.

- **Alignment.** It is based on the Cucker-Smale interaction rule, **BUT** *with a "vision cone" for every bird, so that interactions with other birds are allowed only when they are in the vision cone.*

- **Boundary effect.** *A fast bird close to the edge of the swarm will correct (via a "turning force") to stay inside the swarm.*

A refined Cucker-Smale model

Our particle system is:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = F_i(t) + R_i(t) + A_i(t) + B_i(t) \end{cases}$$

where $F_i(t), R_i(t), A_i(t), B_i(t)$ are respectively the friction, repulsion, alignment and boundary forces exerted on bird i at time t.

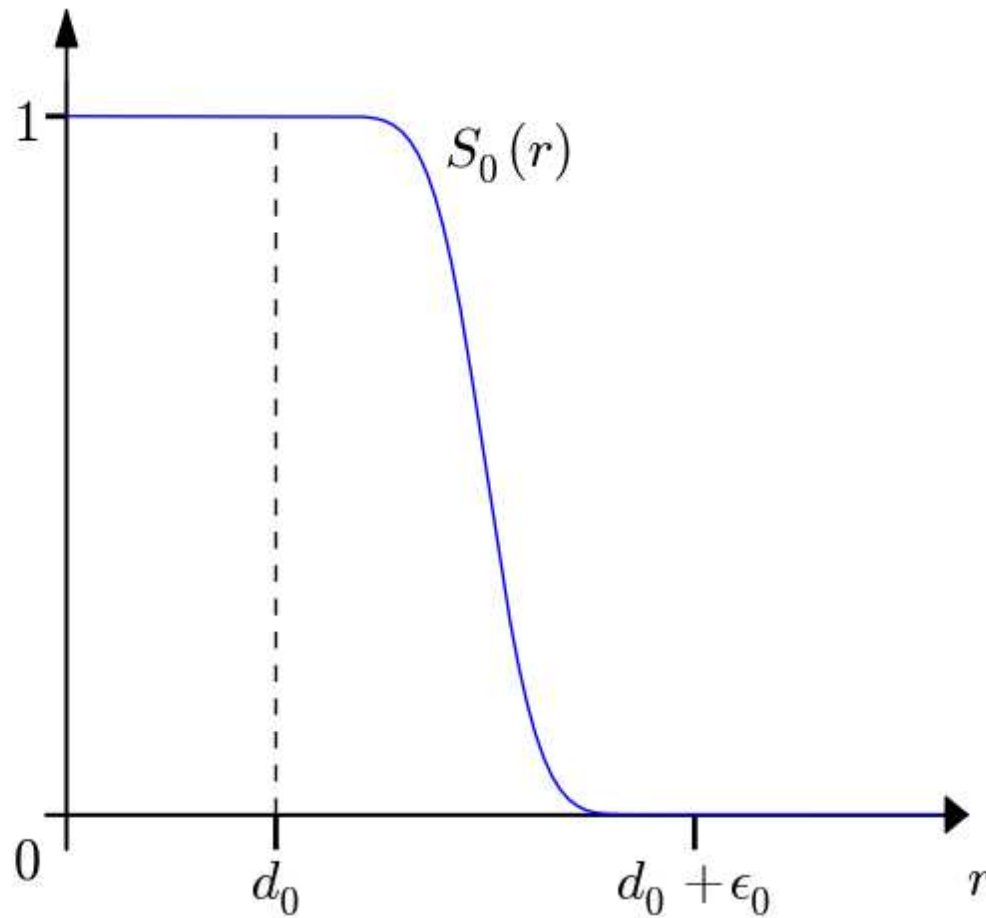- **Friction force** (*always active*). $F_i(t) = \left(\alpha - \beta |v_i(t)|^2 \right) v_i(t)$.

- **Repulsion force.** Active when the distance $|x_i - x_j| \leq d_0$.
  $R_i(t) = \frac{\rho_1}{N} \sum_{j=1}^{N} S_0 \left(|x_i(t) - x_j(t)| \right) \frac{x_i(t) - x_j(t)}{(1 + |x_i(t) - x_j(t)|^2)^{\beta_1}}$
  where S_0 is a (smooth) cutoff function; $S_0(r) = 1$ if $r \leq d_0$ and $S_0(r) = 0$ if $r \geq d_0 + \epsilon_0$, with $d_0, \epsilon_0, \beta_1, \rho_1 > 0$.

The cutoff function S_0

The alignment/flocking force

It is active if the distance between 2 birds is larger than d_0. Denote by α_1 the opening angle of the vision cone for each bird, and set $\delta_1 = \cos(\alpha_1)$.
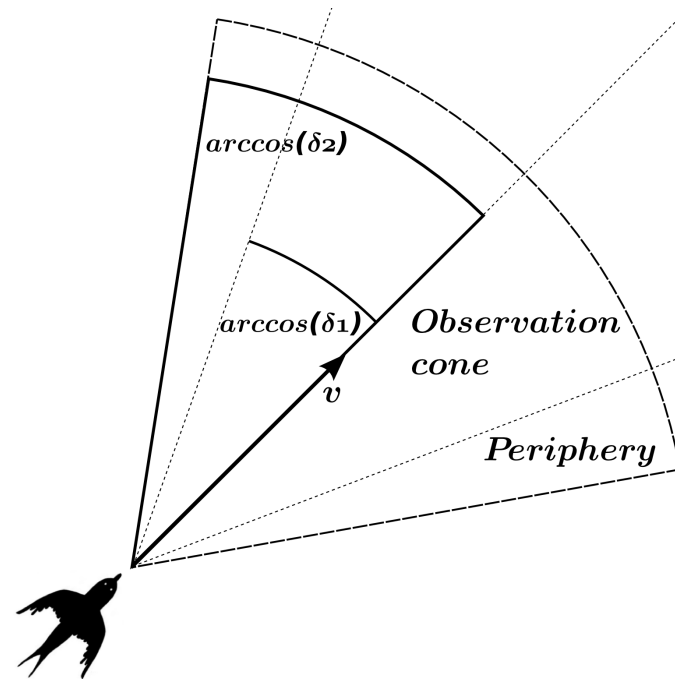
- **Tentative definition.**

$$\tilde{A}_i(t) = \frac{1}{N} \sum_{j=1}^{N} \left[1 - S_0\left(|x_i - x_j|\right)\right] \tilde{W}(x_i - x_j, v_i)(v_j - v_i)$$

where

$$\tilde{W}(x - y, v) = \frac{\gamma}{(1 + |x - y|^2)^{\sigma}} S_2\left(\frac{(y - x) \cdot v}{|y - x||v|}\right)$$

and $S_2(\delta)$ is a (smooth) cutoff function approximating $\chi_{[\delta_1, 1]}$ on $[-1, 1]$, and $\delta = \cos(\alpha)$.

The vision cone

The interpretation of the cutoff function $S_2(\delta)$ is that while the angle $\angle(y - x, v)$ increases from $\arccos(\delta_1)$ to $\arccos(\delta_2)$, bird j is seen in the peripheral vision of bird i, which gradually diminishes as the angle increases and is zero once the bird is outside the cone with angle $\arccos(\delta_2)$.

The alignment/flocking force

- **Modified defintion.** Note that if $v = 0$ (*unrealistic! but mathematically possible*), then $S_2 \left(\frac{(y-x)\cdot v}{|y-x||v|} \right)$ is not well-defined. Therefore, in the rare scenario when $v = 0$, we introduce a "speed cutoff" function $S_1(|v|)$ much like $S_0(r)$, s.t. $S_2 \left(\frac{(y-x)\cdot v}{|y-x||v|} \right)$ applies only when $|v|$ is larger than a minimum speed (i.e., $|v| > d_1$ for some $d_1 > 0$). Then the alignment/flocking force becomes:

$$A_i(t) = \frac{1}{N} \sum_{j=1}^{N} \left[1 - S_0 \left(|x_i - x_j| \right) \right] W(x_i - x_j, v_i)(v_j - v_i),$$

where

$$W(x - y, v)$$

$$= \frac{\gamma}{(1+|x-y|^2)^\sigma} \left\{ S_1(|v|) + [1 - S_1(|v|)] S_2 \left(\frac{(y-x)\cdot v}{|y-x||v|} \right) \right\}$$

and $S_1(|v|) = 1$ if $|v| \le d_1$, $S_1(|v|) = 0$ if $|v| \ge d_1 + \epsilon_1$.

The boundary/turning force

For a fast bird on the edge of the swarm and flying outward, W will be negligibly small, so this bird will simply continue to fly outward and not change speed or direction unless overtaken by other birds. In reality, birds "facing a void" in this way will make an effort to stay with the swarm.

We model this by requiring that such a bird experiences a "*turning*" force *that changes its direction*; and this force smoothly increases with growing "loneliness" *measured by the distance between the bird and others*. This effect is important to keep the swarm together. Assume we are in 3-D and include gravity as a guiding force, pointing in the direction $-k = \langle 0, 0, -1 \rangle$.

$$B_i(t) = C S_3 \left(\rho_i(t) \right) \left(v_i(t) \times \mathbf{k} \right), \quad \text{where}$$

$\rho_i(t) = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{1 + |x_i(t) - x_j(t)|^2}$ measures "*loneliness*", $C > 0$, and S_3 is a smooth cutoff function much like S_0.

Why so many smooth cutoffs?

- It is likely to be realistic; *in reality, transitions such as peripheral vision and the drive to turn around would not happen abrutly, but according to some smooth transition laws*.

- Our analysis requires it; *otherwise, we would face systems of ordinary differential equations with discontinuous right-hand sides. Such ODEs can create unnecessary difficulties at both analytic and numerical levels*.
  Elementary example: $\dot{x}(t) = 1 - 2\chi_{\{x(t)>1\}}, \quad x(0) = 0.$

- They are needed to study well-posedness and stability analysis for swarms of any size; *e.g., following the classical proof of well-posedness for Vlasov-like kinetic equations ([Dobrushin, 1979], [Neunzert, 1984, 1997], ...), one requires some Lipschitz properties on the right-hand sides of the ODEs.*

Velocity bounds

It is not possible to make assertions about the asymptotic behavior of the swarm (*this is really the point!*). But some things are easily seen. Ignore the repulsive force (i.e., set $S_0 = 0$).

Theorem 1 *Let $R(t) = \max_{i=1,\cdots N} |v_i(t)|$ and assume*

$R(0) \geq \sqrt{\frac{\alpha}{\beta}}$. *Then $R(t) \leq R(0)$ for all $t > 0$, that is, the velocity*

support of the swarm stays bounded by $R(0)$.

Proof: Assume w.l.o.g. that $R(t) = |v_1(t)| \geq \sqrt{\frac{\alpha}{\beta}}$ for some

$[a, b] \ni t$. By Cauchy-Schwarz inequality, $v_j \cdot v_1 - v_1^2 \leq 0$, so that

$$\frac{1}{2}\frac{dR^2(t)}{dt} = v_1 \cdot \dot{v}_1$$
$$= (\alpha - \beta|v_1|^2)v_1^2 + \frac{1}{N}\sum_{j=1}^{N} W(x_1 - x_j, v_1)(v_j - v_1) \cdot v_1 + B_1 \cdot v_1.$$

Then $\frac{dR^2(t)}{dt} \leq 0$ because $B_1 \cdot v_1 = 0$ and $(v_j - v_1) \cdot v_1 \leq 0$.

Kinetic equations

Why kinetic equations?

They are useful when the number of birds is very large.

- For large swarms (*i.e., for N very large*), individual birds carry little "weight".

- In reality (?) what matters in this case is the local density of birds in the flock.

- Mathematically, it is often convenient to deal with one continuous equation (*the kinetic equation*) than a large system of ODEs (*the particle system*).

- This translates into a mathematical "stability" question:
  Is the swarm behavior stable as the number of birds grows very large?

Associated kinetic equation

Let $f(t, x, v)$ be the density of birds at (t, x, v). Then $f(t, x, v)$ solves the kinetic equation

$$\partial_t f + v \cdot \nabla_x f = -\nabla_v \cdot \left\{ \left[(\alpha - \beta |v|^2) v + \xi(f) \right] f \right\}$$

where $\xi(f) = \xi_R(f) + \xi_A(f) + \xi_B(f)$, and
$\xi_R(f)(t, x) = \rho_1 \int \int \frac{x-y}{(1+|x-y|^2)^{\beta_1}} S_0(|x - y|) f(y, w, t) dw dy,$

$$\xi_A(f)(t, x, v) = \int \int W(x-y, v)(w-v)(1-S_0(|x-y|)) f(y, w, t) dw dy$$

$$\xi_B(f)(t, x, v) = C S_3(\rho(x, t))(v \times \mathbf{k}) \quad \text{with}$$
$$\rho(x, t) = \int \int \frac{f(y, w, t)}{1+|x-y|^2} dw dy$$

Well-posedness

For the particle system: *ODEs with smooth r.h.s.*

Theorem 2 *(kinetic equation). Assume that $f_0(x,v)$ is compactly supported. Then the kinetic equation has a unique weak solution $f(t)$ with compact support for all $t \in [0,T]$ and for some $T > 0$, s.t. $f(t=0) = f_0$.*

This solution is constructible via a fixed point argument in the space $C\left([0,T], \mathcal{P}_c(\mathrm{IR}^6)\right)$ (of continuous functions in t with values in the compactly supported probability densities on IR^6) equipped with the L^1-Wasserstein metric

$$W_1(f,g) := \sup_{t \in [0,T]} W_1\left(f(t), g(t)\right), \quad \forall f, g \in C\left([0,T], \mathcal{P}_c(\mathrm{IR}^3 \times \mathrm{IR}^3)\right)$$

• The theorem applies, in general, to compactly supported probability measures μ_0 and μ_t.

Stability

The solution of the kinetic equation depends continuously on the initial datum:

Theorem 3 *If $\mu_0, \nu_0 \in \mathcal{P}_c(\mathbb{R}^6)$, and $\mu_t, \nu_t \in C\left([0, T], \mathcal{P}_c(\mathbb{R}^6)\right)$ are resp. the solutions of the kinetic equation starting from μ_0 and ν_0, then there exists a continuous function $r : [0, T] \to \mathbb{R}_+$ with $r(0) = 1$, s.t.*

$$W_1\left(\mu_t, \nu_t\right) \le r(t) W_1\left(\mu_0, \nu_0\right)$$

- In particular, the particle system converges to the kinetic equation as $N \to \infty$.

- It follows that the behavior of a very large flock of birds can (in principle, and on finite intervals) be simulated by solving a system of ODEs for a reasonably large number of birds.

Numerical simulations

Thank you!