Bourgain-Delbaen $\mathcal{L}^\infty$ sums of Banach spaces

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Quasi Prime Banach Spaces

- A Banach space is said to be prime if it is isomorphic to each one of its infinite dimensional complemented subspaces.
- A. Pelczynski The spaces ℓ_p, for $1 \leq p < \infty$ and c_0 are prime spaces.
- J. Lindenstrauss has shown that ℓ_∞ is prime.
- A wider class is that of primary Banach spaces, that have the property that whenever $X \simeq Y \oplus Z$, then either $Y \simeq X$ or $Z \simeq X$.
- Some examples of primary spaces are $C[0,1]$, $L^p(0,1)$.

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Definition

A Banach space X is said to be quasi prime if there exists a subspace Y of X such that X admits a unique non trivial decomposition as $Y \oplus X$. In the case that Y is not isomorphic to X then X is called strictly quasi prime.

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- The authors proved the existence of certain strictly quasi prime Banach spaces in the following result.

Theorem (S.A. Argyros and Th. Raikoftsalis)

The following holds:

For every $1 \leq p < \infty$ there exists a Banach space $\mathfrak{X}_p$ which is strictly quasi prime and admits ℓ_p as a complemented subspace.

There exists a strictly quasi prime Banach space $\mathfrak{X}_0$ containing c_0 as a complemented subspace.

- Each space $\mathfrak{X}_p$, $\mathfrak{X}_0$ is a new type of Schauder sum of a sequence of Banach spaces, the HI Schauder sums that were introduced by S.A.Argyros and V. Felouzis.

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HI- Schauser sums

- We recall that if $(X, \|\cdot\|_*)$ is the Schauder sum of a sequence of Banach spaces $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$, denoted as $X = (\sum_{n=1}^{\infty} \oplus X_n)_*$, then
- There exist bounded projections $P_{[1,n]} : X \to X$ such that $x = \lim_{n\to\infty} P_{[1,n]}(x)$ for every $x \in X$.
- For any element $x \in X$, we define the range of x, $\operatorname{ran} x$, as the minimal interval L of $\mathbb{N}$ such that $x \in \sum_{n\in L} \oplus X_n$.
- We also say that a sequence $(x_k)_{k\in\mathbb{N}}$ in X is horizontally block, if the $\operatorname{ran} x_k < \operatorname{ran} x_{k+1}$ (i.e. $\max \operatorname{ran} x_k < \min \operatorname{ran} x_{k+1}$) for every $k \in \mathbb{N}$.
- The Schauder sum X is shrinking if for every $x^* \in X^*$ $x^* = \lim_{n\to\infty} x^* \circ P_{[1,n]}$.

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- S.A. Argyros and V. Felouzis using a Gowers Maurey type norm proved the following:

Theorem

Let $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$ be a sequence of separable Banach spaces. Then, there exists a Banach space $\mathfrak{X} = (\sum_{n=1}^{\infty} \oplus X_n)_{gm}$ satisfying the following properties:

- *The space $\mathfrak{X}$ is the shrinking Schauder sum of the sequence $(X_n, \|\cdot\|)_{n\in\mathbb{N}}$*
- *Every horizontally block sequence $(x_n)_{n\in\mathbb{N}}$ generates an HI subspace.*

- A Banach space X is HI (Hereditarily indecomposable), if for every closed infinite dimensional subspace Y of X there do not exist closed infinite dimensional subspaces Y_1, Y_2 of Y such that $Y = Y_1 \oplus Y_2$.

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- Each space $\mathfrak{X}_p$, (resp. $\mathfrak{X}_0$) of the Argyros-Raikoftsalis result is the HI-Schauder sum of the corresponding ℓ_p (resp. c_0).
- Moreover, they investigated the finite powers of these spaces they proved:

Theorem (S.A. Argyros-Th. Raikoftsalis)

Let $\mathfrak{X} = \mathfrak{X}_p$ or $\mathfrak{X}_0$ and denote for each $n \in \mathbb{N}$ by $\mathfrak{X}^n$ the space $\sum_{i=1}^{n} \oplus \mathfrak{X}(i)$ endowed with the supremum norm as an external one. Then, for every $n, m \in \mathbb{N}$ with $n \neq m$, the space $\mathfrak{X}^n$ is not isomorphic to $\mathfrak{X}^m$. Moreover, the space $\mathfrak{X}^n$ has at least $n + 1$, up to isomorphism, complemented subspaces.

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- It is open if the aforementioned space $\mathfrak{X}^n$ has exactly $n+1$, up to isomorphism complemented subspaces.
- The above result is a consequence of studying the operators acting on the Gowers- Maurey type HI Schauder sum.

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- We say that an operator S on $\mathfrak{X}$ is horizontally strictly singular if the restriction on an arbitrary (horizontally) block subspace of $\mathfrak{X}$ is not an isomorphism.
- Since every horizontally block sequence in $\mathfrak{X}_p$ is HI it is clear that the space that it generates is totally incomparable to ℓ_p and similarly every subspace of $\mathfrak{X}_0$ generated by a horizontally block subspace is totally incomparable to c_0.
- Therefore, the bounded and linear operator acting on $\mathfrak{X} = \mathfrak{X}_p$ or $\mathfrak{X}_0$ satisfy the property stated on the above theorem.

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- On the Gowers-Maurey space $\mathfrak{X}_{gm}$ every bounded and linear operator is a strictly singular perturbation of a scalar multiple of the identity.
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- The main idea of this work is to construct for every $n \in \mathbb{N}$, a Banach space $\mathcal{Z}^n$ that has exactly $n+1$ complemented subspaces.

- We must mention that W.T. Gowers and B. Maurey proved a similar result using advanced tools, like K-Theory. In particular,

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- We want to have a more straightforward approach, motivated by the Argyros Raikoftsalis result. Namely, the main idea is to construct for a given sequence $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$ of separable Banach spaces, a Banach space $\mathcal{Z}$ such that

(1) $\mathcal{Z} = (\sum_{n=1}^{\infty} \oplus Z_n)_*$, where each Z_n is an augmentation of X_n.

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- A bounded and linear operator K on $\mathcal{Z}$ is called horizontally compact if for every bounded block sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathcal{Z}$, with respect to $(Z_n)_{n\in\mathbb{N}}$, $\|K(z_n)\| \to 0$.
- Equivalently, for every $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$, such that $\|K \circ P_{(k_\varepsilon,\infty)}(x)\| < \varepsilon\|x\|$ for every $x \in \mathcal{Z}$.
- The second condition that we want concerning the operators acting on $\mathcal{Z}$, is stronger than the corresponding of the Gowers-Maurey HI-Schauder sums.
- The finite powers of such a space $\mathcal{Z}$, for a specifically chosen sequence $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$ could satisfy the desired result.

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There exists a hereditarily indecomposable Banach space $\mathfrak{X}_k$ with the "scalar-plus-compact" property.

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- The definition of BD-$\mathcal{L}^\infty$-Sums of Banach spaces uses the original BD- construction.
- Let $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$ be a sequence of separable Banach spaces. We say that a Banach space $\mathcal{Z}$ is a BD-$\mathcal{L}^\infty$-sum of $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$ if there exists a sequence $(\Delta_n)_{n\in\mathbb{N}}$ of finite, pairwise disjoint subsets of $\mathbb{N}$ and the following are satisfied:
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- The sets $(\Delta_n)_{n\in\mathbb{N}}$ are defined recursively following the BD-method.
- Each element $\gamma \in \Delta_k$ is determined by a functional $c_\gamma^* : (\sum_{n=1}^{k-1} \oplus (X_n \oplus \ell^\infty(\Delta_n)))_\infty \to \mathbb{R}$.
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- $i_{l,k} = i_{l,m} \circ i_{m,k}$ for every $l \leq m \leq k$ and

- $i_{k-1,k}(x) = x$ for every $x \in \sum_{n=1}^{k-1} \oplus X_n$, while

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- The BD- method yields that $i_{k,m}$ are uniformly bounded by a constant $C > 0$ and therefore we can define $i_k = \lim_{m\to\infty} i_{k,m}$.
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Let $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$ be a sequence of separable Banach spaces. Then there exists a Banach space $\mathcal{Z}$ with the following properties:

- *$\mathcal{Z}$ is the BD-$\mathcal{L}^\infty$ sum of $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$,*
- *$\mathcal{Z}$ admits a shrinking Schauder decomposition, $\mathcal{Z} = \sum_{k=1}^\infty \oplus Z_k$.*
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- *$\mathcal{Z}^*$ may be identified with $(\sum_{n=1}^\infty \oplus Z_n^*)_1$.*

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Let $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$ be a sequence of separable Banach spaces. Then there exists a Banach space $\mathcal{Z}$ with the following properties:

$\mathcal{Z}$ is the BD-$\mathcal{L}^\infty$ sum of $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$,

$\mathcal{Z}$ admits a shrinking Schauder decomposition, $\mathcal{Z} = \sum_{k=1}^{\infty} \oplus Z_k$.

Every horizontally block sequence $(z_n)_{n\in\mathbb{N}}$ generates an HI subspace.

$\mathcal{Z}^*$ may be identified with $(\sum_{n=1}^{\infty} \oplus Z_n^*)_1$.

AH-$\mathcal{L}^\infty$ sums

- The properties of $\mathcal{Z}$ are strongly based on the existence of special features that are preserved by the Argyros-Haydon HI method of construction.
- We denote by $\mathcal{Z}_p$ (resp. $\mathcal{Z}_0$) the AH-$\mathcal{L}^\infty$ sum of the corresponding ℓ_p (resp. c_0).
- Then, $\mathcal{Z}_p = \sum_{k=1}^{\infty} \oplus Z_k$, where $Z_k = i_n[\ell_p \oplus \ell^\infty(\Delta_k)]$.
- Each Z_k is isomorphic to $(\ell_p \oplus \ell^\infty(\Delta_k))_\infty$ which is C_k-isomorphic to ℓ_p with $C_k \to \infty$. Therefore, we cannot have an isometry.
- For every $k \in \mathbb{N}$, $P_{[1,k]}(\mathcal{Z}_p) \simeq \ell_p$ and $P_{[1,k]}(\mathcal{Z}_0) \simeq c_0$.

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Quasi Prime AH-$\mathcal{L}^\infty$ sums

- The following are proved:
- For every $1 \leq p < \infty$ the space $\mathfrak{Z}_p$ is strictly quasi prime and admits ℓ_p as a complemented subspace.
- The space $\mathfrak{Z}_0$ is strictly quasi prime containing c_0 as a complemented subspace.
- Let $\mathfrak{Z} = \mathfrak{Z}_p$ or $\mathfrak{Z}_0$. Then, for every bounded and linear operator T on $\mathfrak{Z}$, there exists scalar λ such that $T - \lambda I$ is horizontally compact.

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- In terms of studying the operators acting on $\mathcal{Z}_p$ and $\mathcal{Z}_0$ we use a special type of block sequences, the Rapidly Increasing sequences (RIS). Following the AH -method of construction we prove the following:
- Let $\mathcal{Z} = \mathcal{Z}_p$ or $\mathcal{Z}_0$.
- Let Y is a Banach space and $T : \mathcal{Z} \to Y$ is a bounded and linear operator such that $\|T(x_n)\| \to 0$ for every RIS $(x_n)_{n\in\mathbb{N}}$, then $\|T(x_n)\| \to 0$ for every bounded (horizontally) block sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{Z}$.
- If $T : \mathcal{Z} \to \mathcal{Z}$ is a linear and bounded operator, then $\text{dist}(Tx_n, \mathbb{R}x_n) \to 0$ for every RIS $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{Z}$.

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The Main Result

- Studying the finite powers of $\mathcal{Z} = \mathcal{Z}_p$ or $\mathcal{Z}_0$ we prove

Theorem

The space $\mathcal{Z}^n = \sum_{i=1}^{n} \oplus \mathcal{Z}$ endowed with the external supremum norm, we prove admits $n+1$- pairwise not isomorphic complemented subspaces.

- As in the Argyros- Raikoftsalis construction, we already have that $\mathcal{Z}^n$ is not isomorphic to $\mathcal{Z}^m$ for every $n \neq m$ which implies that $\mathcal{Z}^n$ has at least $n+1$, pairwise not isomorphic complemented subspaces.

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- We prove that the matrix $\Lambda = (\lambda_{i,j})_{1 \le i,j \le n}$ is a projection on $\mathbb{R}^n$ and let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an inventible matrix of the form $A = (a_{i,j})_{1 \le i,j \le n}$ such that $A \Lambda A^{-1} = (\tilde{\lambda}_{i,j})_{1 \le i,j \le n}$ with $\tilde{\lambda}_{i,j} = \begin{cases} 0, & \text{if } i \neq j \\ 0 \text{ or } 1, & \text{if } i = j. \end{cases}$
- Considering the inventible operator $\tilde{A} = (a_{i,j} I_{i,j})_{1 \le i,j \le n}$ on $\mathcal{Z}_p^n$, we set $\tilde{P} = \tilde{A} P \tilde{A}^{-1}$ and the following hold:

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- Thus, for every $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that $\|\tilde{K}_{i,j} \circ P_{(k_\varepsilon,\infty)}|_{\mathcal{Z}_{p(j)}}\| < \varepsilon$ for every i, j.
- Setting $L = \{i : \tilde{\lambda}_{i,i} \neq 0\}$, we show that $L \neq \emptyset$ and $W \simeq (\sum_{i \in L} \oplus \mathcal{Z}_p) \oplus Y$, where $Y \simeq \ell_p$.
- Since $\mathcal{Z}_p \simeq \mathcal{Z}_p \oplus \ell_p$, the result follows.

Complemented subspaces of Z_p^n

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- Thus, for every $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that $\|\tilde{K}_{i,j} \circ P_{(k_\varepsilon,\infty)}|_{\mathcal{Z}_{p(j)}}\| < \varepsilon$ for every i, j.
- Setting $L = \{i : \tilde{\lambda}_{i,i} \neq 0\}$, we show that $L \neq \emptyset$ and $W \simeq (\sum_{i \in L} \oplus \mathcal{Z}_p) \oplus Y$, where $Y \simeq \ell_p$.
- Since $\mathcal{Z}_p \simeq \mathcal{Z}_p \oplus \ell_p$, the result follows.

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Thank You!