Closed ideals of operators on Banach spaces

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(joint work with N J Laustsen, E Odell, Th Schlumprecht)

Banach space theory workshop, BIRS, 5–9 March 2012

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E.g., $X = \left(\bigoplus_{n=1}^{\infty} \ell_1^n \right)_{c_0}$, $X = C(K)$, etc.

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$$T^{(m)}\colon \Big(\bigoplus_{n\in R_m} \ell_1^n\Big)_{\ell_\infty} \to \ell_1^m$$

for some finite set $R_m \subset \mathbb{N}$.

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(ii) Assume that for each $m \in \mathbb{N}$ we have $X_m = \ell_1^{N_m}$ for some $N_m \in \mathbb{N}$. If the T_m have uniform approximate lattice bounds, then they uniformly approximately factor through ℓ_∞^n's.

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- Not even for $X_m = \ell_\infty^{N_m}$!

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Denote by $e_{ij} = e_{ij}^{(m)}$ the unit vector basis of $\ell_\infty^m(\ell_1^m)$.

The norm of $\sum_{ij} a_{ij} e_{ij}$ is given by $\max_i \sum_j |a_{ij}|$.

We let $T_{ij}^{(m)} = T^{(m)}(e_{ij})$ and identify $T^{(m)}$ with the $m \times m$ matrix $\left(T_{ij}^{(m)}\right)$ in L_1.

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(ii) or the $T^{(m)}$ uniformly approximately factor through ℓ_∞^k's.

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