Strictly singular non-compact operators on a class of HI spaces

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joint work with A. Pelczar-Barwacz.

Question : find a strictly singular (ss) non-compact operator on the HI spaces defined having as frame either Schlumprecht space or Argyros-Deliyanni mixed Tsirelson spaces.

- G. Androulakis- Th. Schlumprecht: There exist ss and non-compact on W.T. Gowers-B.Maurey space.
- I. Gasparis. There exist ss non-compact operators on the HI spaces based in the mixed Tsirelson spaces $T[\mathcal{S}_n, \theta_n]$ "assuming" the existence of c_0^ω-spreading model in the dual space.
- Gasparis method was adapted by,
- Argyros-Deliyanni-Tolias for constructing HI spaces with diagonal strictly singular non-compact operators,
- K.Beanland, for constructing non-trivial strictly singular operators on asymptotic ℓ_p HI spaces.

Coding function

Let $\mathcal{F}_n = \mathcal{A}_n (= \{F \subset \mathbb{N} : \#F \leq n\})$ or $\mathcal{F}_n = \mathcal{S}_n$ the nth-Schreier family for every n.
Let

$$\mathcal{W} = \{(f_1, \ldots, f_k) : \; f_1 < \cdots < f_k \in c_{00}(\mathbb{Q}), \|f_i\|_\infty \leq 1, \; k \in \mathbb{N}\}$$

Fix an injective function $\sigma : \mathcal{W} \to \mathbb{N}$.
Let $(D_n)_n$ be a sequence of families of finite subsets of $\mathbb{Q}$.
A block sequence $(f_1, \ldots, f_k)$ is $(\sigma, \mathcal{F}_n)$-admissible w.r. the sets $(D_n)_n$,

1. if $(f_1, \ldots, f_k)$ is $\mathcal{F}_n$-admissible,
2. $f_1 \in \bigcup_n D_n$ and $f_{i+1} \in D_{\sigma(f_1, \ldots, f_i)}$ for any $i < k$.

Definition of the norming set

Let $1 > (\theta_n)_{n\in\mathbb{N}} \searrow 0$.
Fix $L \subset \mathbb{N}$ and $1 > (\rho_l)_{l\in L} \searrow 0$ such that $\rho_l \geq \theta_l$ for any $l \in L$.
Let σ be a coding function.
For any $D \subset c_{00}(\mathbb{Q})$ define for $n \in \mathbb{N}$ and $l \in L$,

$$D_n = \{\theta_n \sum_{i=1}^{k} f_i : \ f_1, \ldots, f_k \in D, \ (f_1, \ldots, f_k) \ \mathcal{F}_n\text{-admissible}, \ k \in \mathbb{N}\},$$

$$D_l^{\sigma} = \{\rho_l \sum_{i=1}^{k} Ef_i : E \subset \mathbb{N} \,\text{interval}, \ (f_i)_{i=1}^{k} \subset D \ (\sigma, \mathcal{F}_l)\text{-admissible w.r.} \,(D_n)_n\}.$$

The elements $\cup_l D_l^{\sigma}$ are called special functionals.
For $f \in D_n$ we set $w(f) = \theta_n$ and for $f \in D_l^{\sigma}$, $w(f) = \rho_l$.

Consider a symmetric subset $D \subset c_{00}(\mathbb{Q})$ such that

1. $(\pm e_n^*)_n \subset D$,
2. $D_n \subset D$ for any $n \in \mathbb{N}$.
3. $D \subset \bigcup_{n \in \mathbb{N}} D_n \cup \bigcup_{l \in L} D_l^\sigma$,

Take X_D be the completion of $(c_{00}, \|\cdot\|_D)$ where

$$\|x\|_D = \sup\{f(x) : f \in D\}.$$

We set X_u be the space defined for $D = \cup_n D_n$.

Properties of the spaces X_D, X_u.

1) $\|x\|_u \leq \|x\|_D$
2) The spaces X_D, X_u are reflexive.
3) (e_n) is bimonotone a basis for X_D and unconditional basis for X_u.
4) The basis of X_D and X_u are asymptotically equivalent i.e. [AS] for the families $\mathcal{A}_n$, the spreading model of the basis of X_D is the basis of X_u,
   -[P] for the families $\mathcal{S}_n$ it means that are $\mathcal{S}_\omega$-equivalent i.e there is $C \geq 1$ and an increasing sequence $(i_n) \subset \mathbb{N}$ such that for any n and $i_n \leq F \in \mathcal{S}_n$ the sequences $(e_i)_{i \in F}$ in X_u and $(e_i)_{i \in F}$ in X_D are C-equivalent.

- Schlumprecht's space is the space $T[(\mathcal{A}_n, 1/\log_2(n+1))_n]$ taking $D_l^\sigma = \emptyset$.
  Defining the sets D_l^σ we have Gowers-Maurey space .
- Argyros-Deliyanni mixed Tsireslon spaces are the spaces $T[(\mathcal{S}_n, \theta_n)_n]$ taking $D_l^\sigma = \emptyset$
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Result

Theorem

For the space X_D defined by the families $\mathcal{S}_n$, there exists a bounded, strictly singular, non-compact

$$T : X_D \to X_D$$

provided that there exists $c > 0$ such that $\lim_n \frac{\theta_{n+m}}{\theta_n} > c$ for every m.

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We shall present the proof for the spaces defined by the Schreier families $(S_n)_n$. The strictly singular non-compact operator will be

$$T(x) = \sum_{n=1}^{\infty} f_n(x) e_{k_n}$$

for an appropriate sequence of seminormalized functionals $(f_n)_n \subset X_D^*$.
Our proof inspired by
1) the idea of Androulakis-Schlumprecht, to have an "infinite tree" construction which determines the functionals $(f_n)_n$.
2)[ADT] Let X, Y be Banach spaces such that

a) there exists $(x_n^*)_n \subset B_{X^*}$ generating c_0-spreading model.
b) Y has normalized basis and there exists norming set D of Y such that for all $\varepsilon > 0$ there exists $M_\varepsilon \in \mathbb{N}$ such that for all $f \in D$

$$\#\{n : |f(e_n)| > \varepsilon\} \leq M_\varepsilon.$$

Then $T : X \to Y, T(x) = \sum_n x^*_{q_n}(x) e_n$ is bounded non-compact, for appropriate $(q_n)_n$

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The basic ingredients

$x = \sum_{i \in F} a_i e_i$ is (n, ε)-basic special convex combination (scc) if

$$F \in \mathcal{S}_n, \ \sum_{i \in F} a_i = 1 \ \text{and} \ \sum_{i \in G} a_i < \varepsilon \ \forall G \in \mathcal{S}_{n-1}.$$

For an (n, ε)-basic scc it holds

$$1 \leq \|\theta_n^{-1} x\| \leq 1 + \varepsilon.$$

If $(x_i)_{i \in F}$ is a block sequence, $x = \sum_{i \in F} a_i x_i$ is said to be (n, ε)-special convex combination (scc) if $\sum_{i \in F} a_i e_{\text{maxsupp}\, x_i}$ is (n, ε)-basic scc.

In the sequel we shall omit the numbers ε.

Periodic RIS

Let $n_0, M \in \mathbb{N}$ and $n_1, \ldots, n_M \in \mathbb{N}$.
Let $x_{(i-1)M+j}, i \leq N, j \leq M$ be a block sequence such that

1) For every $i \leq N$, $x_{(i-1)M+j}$ is a seminormalized n_j-basic scc. i.e.
$$x_{(i-1)M+j} = \theta_{n_j}^{-1} \sum_{k \in F_{i,j}} a_k e_k$$

2) $x = \sum_{i=1}^{N} \sum_{j=1}^{M} a_{(i-1)M+j} x_{(i-1)M+j}$ is an $n_0 - scc$.

a) We call the vector x an (n_0, M)-periodic average.

b) Taking $(n_j)_{j=1}^{M}$ "very fast increasing" we call the sequence $(x_{(i-1)M+j})_{i,j}$, (n_0, M)-periodic rapidly increasing sequence (RIS) of height 1.

- For $N = 1$ and $(n_j)_j$ appropriate chosen, we have the notion of rapidly increasing sequence of lenght M.
- Vectors similar to periodic averages have used by D. Leung, W-K Tang to provide examples of mixed Tsireslon spaces $T[(\mathcal{S}_n, \theta_n)_n]$ not isomorphic to their modified version.

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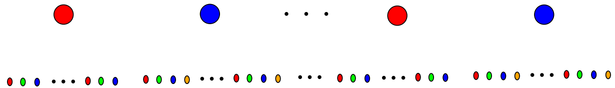
Let $x^1 = \sum_{i=1}^{N}\sum_{j=1}^{M} a_{(i-1)M_0+j}x^1_{(i-1)M_0+j}$ (n_0, M)-periodic RIS of height 1.
We get periodic RIS of height 2 by gluing periodic RIS of height 1 and preserving the characteristics numbers $M, n_0, n_1, \dots, n_M$ of x^1.



The first line is $(n_0, 2)$-periodic RIS of height 1. We take two averages with admissibility's $n_1 << n_2$. We take repetitions of the two "nodes" to have n_0-admissibility.
We get periodic average of height 2, by "substituting "

1. the first node by an (n_1, M_1)-periodic RIS, with $M_1 >> M$ different admissibility's, $n_{1,1}, n_{1,2}, \dots, n_{1,M_1}$, and n_1-admissibility
2. and the second node by an (n_2, M_2)-periodic RIS, with $M_2 >> M_1$ different admissibility's, $n_{2,1}, n_{2,2}, \dots, n_{2,M_2}$, and n_2-admissibility
3. Moreover we take much more repetitions of these two nodes in order to have again S_{n_0}-admissibility.

We continue in the same manner to define periodic RIS of height n. We shall use two trees,

- one that determines the number of the different admissibilities and different $\theta_n's$ that appear in each periodic average,
- the other to index all the elements of the periodic average.

Let $\mathcal{R} \subset \cup_n \mathbb{N}^n$ be an infinite tree with unique root. We set for $\beta \in \mathcal{R}$

- $M_\beta = \# succ(\beta)$ (the number of different admissibilities that appear in each periodic RIS)
- we associate also a parameter m_β (the admissibility of each periodic average)

For each n let $\mathcal{T}_n$ be a tree of height n, $\upsilon : T_n \to \mathcal{R}$ such that for every $\alpha \in \mathcal{T}_n$, not terminal

$$\text{succ}(\alpha) = \{\alpha^\frown((i-1)M_{\upsilon(\alpha)} + j) : i \leq N_\alpha, j \leq M_{\upsilon(\alpha)}\}.$$

Periodic RIS of height n (with tree-analysis)

We say that the vector $x_n \in X$ is periodic average of height n, with tree-analysis determined by the core tree $\mathcal{R}$, if there is a family $(x_\alpha)_{\alpha \in \mathcal{T}_n}$,

1. for any terminal node $\alpha \in \mathcal{T}_n$ we have $|\alpha| = n$ and $x_\alpha = e_{t_\alpha}$ for some $t_\alpha \in \mathbb{N}$,
2. for any node $\alpha \in \mathcal{T}_n$ with $|\alpha| = n-1$ the vector x_α is a seminormalized $m_{\upsilon(\alpha)}$-basic special combination of $(x_\beta)_{\beta \in \text{succ}(\alpha)}$ i.e. $x_\alpha = \theta^{-1}_{m_{\upsilon(\alpha)}} \sum_{i \in F_\alpha} c_i e_i$.
3. for any node $\alpha \in \mathcal{T}$ with $|\alpha| < n-1$ the vector x_α is a seminormalized $(m_{\upsilon(\alpha)}, M_{\upsilon(\alpha)})$-periodic average of $(x_\beta)_{\beta \in \text{succ}(\alpha)}$, i.e.

$$x_\alpha = \theta^{-1}_{m_{\upsilon(\alpha)}} \sum_{k=1}^{N_\alpha} \sum_{j=1}^{M_{\upsilon(\alpha)}} a_{\alpha^\frown((k-1)M_{\upsilon(\alpha)}+j)} x_{\alpha^\frown((k-1)M_{\upsilon(\alpha)}+j)} \qquad (1)$$

Proposition

For appropriate choice of the parameters m_α, M_α of the core tree, it holds that

$$\|x_n\|_D \leq \prod_{i=1}^{n}(1+3\theta_{n_i})$$

We associate to the periodic RIS x_n in a natural way the functional f_n with tree analysis $(f_\alpha^n)_{\alpha\in\mathcal{T}_n}$ where

1. $f_\alpha = e_\alpha^*$ for α terminal.
2. $f_\alpha^n = \theta_{m_{\upsilon(\alpha)}} \sum_{\beta\in\text{succ}(\alpha)} e_\beta^*$ if $|\alpha| = n-1$.
3. $f_\alpha^n = \theta_{m_{\upsilon(\alpha)}} \sum_{\beta\in\text{succ}(\alpha)} f_\beta^n = \theta_{m_{\upsilon(\alpha)}} \sum_{i=1}^{N_\alpha} \sum_{j=1}^{M_{\upsilon(\alpha)}} f_{\alpha\frown((i-1)M_{\upsilon(\alpha)}+j)}$.

The associated functionals $f_n = f_\emptyset^n$ satisfies $f_n(x_n) = 1$ and

$$\prod_{i=1}^{n}(1+3\theta_{n_i})^{-1} \leq \|f_n\| \leq 1.$$

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The functionals f_n will be used to define the operator

$$T(x) = \sum_n f_n(x) e_{i_n}$$

The seminormalization of $f_n's$ yields that T is not compact.

$$\|Tx_n - Tx_m\| = \|f_n(x_n)e_{i_n} - f_m(x_m e_{i_m})\| \geq 1.$$

We shall make carefully choice of the parameters M_γ, $m_\gamma, \gamma \in \mathcal{R}$ to have that T is bounded and strictly singular.
How we choose m_γ?

The choice of the core tree $\mathcal{R}$

We enumerate the nodes of $\mathcal{R}$ as γ_j following the lexicographic order. We refine the choice of $M_{\gamma_i}, m_{\gamma_i}$, which ensure the seminormalization of $x_n' s$, by choosing for the node γ_j a positive integer k_{γ_j} such that

1. $\rho_{k_{\gamma_j}}(\sum_{i<j} m_{\gamma_i} + M_{\gamma_i}) = \varepsilon_{\gamma_j}$ and $\sum_j \varepsilon_{\gamma_j} < 1.$
   (recall that ρ_j are the weights of the special functionals)
2. We choose m_{γ_j} such that $\dfrac{\theta_{m_{\gamma_j}}}{\theta_{m_{\gamma_j}+k_{\gamma_j}+\mathrm{ord}(\gamma_j)}} \leq c^{-1}$.

The last choice is possible by the assumption $\lim_n \dfrac{\theta_{n+m}}{\theta_n} > c$ for every m.
To simplify notation $\rho_{k_{\gamma_j}} = \rho_{k_j}$, $f_{k_{\gamma_{n+1}}} = f_n$, $e_{i_{r_{\gamma_{n+1}}}} = e_{i_n}$.
We show that the operator

$$T(x) = \sum_{n=1}^{\infty} f_n(x) e_{i_n}$$

is bounded and strictly singular.

If we consider the space X_u it follows easily that T is bounded since

$$\|\sum_{n=1}^{\infty} f_n(x) e_{i_n}\|_u \leq \|x\|_u \tag{2}$$

since it holds $\|\sum_n a_n e_{i_n}\| \leq \|\sum_n a_n u_n\|$, $i_n \leq u_n$ is normalized block basis.

Preparatory work

We take any $x \in X_D$ with a finite support and a norming functional f with $f(Tx) = \|Tx\|_D$.

1. It holds that $\forall f \in D, \forall j \in \mathbb{N}$, $\{n : |f(e_n)| > \rho_{k_j}\} \in \mathcal{S}_{k_j}$
2. We partition $\mathbb{N}$ to the sets

$$B_j = \{n \in \mathbb{N} : \ \rho_{k_{j+1}} < |f(e_{i_n})| \leq \rho_{k_j}\} \in S_{k_{j+1}}$$

Let $D_j = B_j \cap \{1, \ldots, \sum_{i<j} M_i + \sum_{i<j} m_i\}$, the "initial" part of B_j.

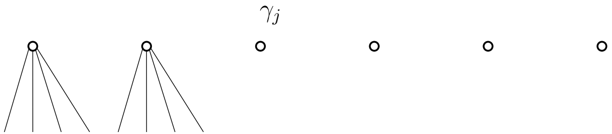
For simplicity assume $D_j = \emptyset$ and $f(e_{i_n}) \geq 0$ Then we have

$$\|Tx\|_D = f(Tx) = \sum_{j=1}^{\infty} \sum_{n \in B_j} f_n(x) f(e_{i_n}).$$

Analyzing the functionals using the tree structure

Set for $\gamma_j \in \mathcal{R}$,

$$I_{\gamma_j} = \{\beta \in \mathcal{R} : |\beta| = |\gamma_j|, \gamma_j <_{lex} \beta\} \cup \cup_{|\beta|=|\gamma_j|, \beta <_{lex} \gamma_j} \mathrm{succ}(\beta)$$



The set I_{γ_j}

We have that for every n,

1. $f_n = \sum_{\beta \in I_j} \sum_{\alpha \in \mathcal{T}_n : \upsilon(\alpha) = \beta} c_\beta f_\alpha^n + \sum_{\alpha \in \mathcal{T}_n : \upsilon(\alpha) = \gamma_j} c_{\gamma_j} f_\alpha^n$.
2. For every $\beta \in \mathcal{R}$ the set $\{f_\alpha^n : \alpha \in \mathcal{T}_n, \upsilon(\alpha) = \beta\} \in \mathcal{S}_{\mathrm{ord}(\beta)}$.

Using the above properties and $\frac{\theta_{m_\beta}}{\theta_{m_\beta+\text{ord}(\beta)+k_\beta}} \leq c^{-1}$ we get that c_0-behavior of the nodes thar are determined by a node β of the core tree

Lemma

For $\beta \in \mathcal{R}$ and for every $F \in \mathcal{S}_{k_\beta}$, $F > |\beta|$

$$\|\sum_{n\in F}\sum_{\alpha\in\mathcal{T}_n,\upsilon(\alpha)=\beta} f_\alpha^n\| \leq c^{-1}. \quad (3)$$

Since for $n \in B_j$, $f_n = \sum_{\beta\in I_j}\sum_{\alpha\in\mathcal{T}_n:\upsilon(\alpha)=\beta} c_\beta f_\alpha^n + u_n$, $u_n = \sum_{\alpha\in\mathcal{T}_n:\upsilon(\alpha)=\gamma_j} c_{\gamma_j} f_\alpha^n$

Corollary

For any $\gamma_j \in \mathcal{R}$ and $F \in \mathcal{S}_{k_{j+1}}$ with $F > |\gamma_j| + 2$,

$$\|\sum_{n\in F}(f_n - u_n)\| = \|\sum_{\beta\in I_j}\sum_{n\in F}\sum_{\alpha:\upsilon(\alpha)=\gamma_j} c_j f_\alpha^n\| \leq \frac{\#I_j}{c}$$

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Corollary

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T is bounded

$$\|Tx\|_D = f(Tx) \leq \sum_{j=1}^{\infty} |\sum_{n\in B_j}(f_n - u_n)(x)f(e_{i_n})| + \|\sum_{j=1}^{\infty} c_j \sum_{n\in B_j} u_n(x)e_{i_n}\|$$

Corollary for $F = B_j$ yields

$$\sum_{j=1}^{\infty} |\sum_{n\in B_j}(f_n - u_n)(x)f(e_{i_n})| \leq \sum_{j=1}^{\infty} \|\sum_{n\in B_j}(f_n - u_n)\| \, \|x\|_D \; \rho_{k_j}$$

$$\leq (\sum_{j=1}^{\infty} \frac{1}{c2^j})\|x\|_D, \text{by the choice of } \rho_{k_j}.$$

It follows

$$\|Tx\|_D \leq (\sum_{j=1}^{\infty} \frac{1}{c2^j})\|x\|_D + \|\sum_{j=1}^{\infty} c_j \sum_{n\in B_j} u_n(x)e_{i_n}\|_D$$

T is bounded, second summand

To estimate $\|\sum_{j=1}^{\infty} c_j \sum_{n\in B_j} u_n(x)e_{i_n}\|_D$ we use the

1. the admissibility of the sets B_j
2. The asymptotic equivalence of the basis of X_D and X_u
3. $\|\sum_n g_n(x)e_{i_n}\|_u \leq \|x\|_u$

So partitioning $j's$ according the predecessor of γ_j we get

$$\|\sum_{j=1}^{\infty} c_j \sum_{n\in B_j} u_n(x)e_{i_n}\|_D \leq \sum_{k=0}^{\infty} c_k\theta_{m_k}\| \sum_{\gamma_j\in \mathrm{succ}(\gamma_k)} \sum_{n\in B_j} u_n(x)e_{i_n}\|_D$$

$$\leq C\sum_{k=0}^{\infty}\theta_{m_k}\| \sum_{\gamma_j\in \mathrm{succ}(\gamma_k)} \sum_{n\in B_j} u_n(x)e_{i_n}\|_u \leq \frac{C}{c}\sum_{k=0}^{\infty}\theta_{m_k}\|x\|_u$$

$$\leq (\frac{C}{c}\sum_{k=0}^{\infty}\frac{1}{2^k})\|x\|_D$$

T is strictly singular

To show that T is strictly singular, we have

$$\|T(x)\|_D \leq \sum_{j=1}^{j_0} |\sum_{n\in B_j} f_n(x) f(e_{i_n})| + \sum_{j>j_0} |\sum_{n\in B_j} f_n(x) f(e_{i_n})|$$

From the proof that T is bounded we get,

$$\sum_{j>j_0} |\sum_{n\in B_j} f_n(x) f(e_{i_n})| \leq \frac{K}{2^{j_0}} \|x\|_D.$$

For the first term we use that the space which is the completion of $c_{00}(\mathbb{N})$ under the norm

$$\|x\|_{j_0} = \sup\{\sum_{n\in F} \varepsilon_n f_n(x) : \varepsilon_n \in \{-1,1\},\ F \in \mathcal{S}_{k_{j_0+1}}\}$$

is c_0-saturated and $\|x\|_{j_0} \leq \theta^{-1}_{k_{j_0+1}} \|x\|_D$. ($X_D^*$ is closed in $(\mathcal{S}_n, \theta_n)$-operations.

X_D is reflexive$\Rightarrow$ every $\varepsilon > 0$ and every subspace Y of X_D there exists $x \in S_{X_D}$ with $\|x\|_{j_0} < \varepsilon$.
It follows,

$$\begin{aligned}\|Tx\|_D &\leq \sum_{j=1}^{j_0} \Big|\sum_{n\in B_j} f_n(x) f(e_{i_n})\Big| + \qquad \sum_{j_0+1}^{\infty} \Big|\sum_{n\in B_j} f_{r_n}(x) f(e_{i_n})\Big| \\ &\leq j_0 \|x\|_{j_0} + \frac{K}{2^{j_0}} \\ &\leq j_0 \varepsilon + \frac{K}{2^{j_0}}\end{aligned}$$

Since this holds for every $\varepsilon > 0$ we get that T is strictly singular.

Thank you