Supports and ranges in Banach spaces

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1. Banach's hyperplane problem, Gowers' dichotomies and classification program
2. Other dichotomies and progress in the classification
   Joint work with C. Rosendal, 2007
3. Properties of Gowers and Maurey's spaces
   Joint work with Th. Schlumprecht, 2011

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Proposition (Casazza, 90's)

A space which satisfies Casazza's criterion is isomorphic to no proper subspaces.

Theorem (Gowers-Maurey, 90's)

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But then Gowers and Maurey improved the properties of *GM*.

Theorem (Gowers-Maurey, 90's)

The space GM is HI and no HI space is isomorphic to its proper subspaces.

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Observation

Let (e_n) be the natural basis of the complex GM space. Then $e_1, ie_1, e_2, ie_2, \ldots$ is an even-odd real basis of GM, yet GM is not $\mathbb{R}$-linearly isomorphic to its real proper subspaces.

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But the problem remained in the complex case.

Actually our results will suggest that *GM* fails Casazza's criterion in a strong way:

Theorem (F., Schlumprecht, 11)

A version of GM is saturated with even-odd block sequences.

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A space is said to be *quasi-minimal* if any two subspaces have further subspaces which are isomorphic.

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Theorem (Gowers' 2nd dichotomy, 02)

Every Banach space contains a quasi-minimal subspace or a subspace with a basis such that no two disjointly supported block subspaces are isomorphic.

Note that the property that no two disjointly supported block subspaces are isomorphic is a strong form of the criterion of Casazza.

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(a) the classes are hereditary, i.e., stable under taking subspaces (or block subspaces),
(b) the classes are inevitable, i.e., every infinite dimensional Banach space contains a subspace in one of the classes,
(c) the classes are mutually disjoint,
(d) belonging to one class gives some information about the operators that may be defined on the space or on its subspaces.

1. Gowers' list of four classes

Finally, H. Rosenthal had defined a space to be *minimal* if it embeds into any of its subspaces. A quasi minimal space which does not contain a minimal subspace was called *strictly quasi minimal* by Gowers.

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Gowers deduced from these dichotomies and from easy implications (e.g. HI implies strictly quasi minimal) a list of four inevitable classes of Banach spaces characterized by the properties:

- HI spaces (GM),
- no disjointly supported subspaces are isomorphic (G_u),
- strictly quasi-minimal with an unconditional basis (T),
- minimal spaces (c_0, ℓ_p, T^*, S).

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2. New dichotomies

The second dichotomy of Gowers is of the form "many versus few" isomorphisms between subspaces. We shall now define another dichotomy of this form.

We use here a presentation of results of F. - Rosendal (2007) based on observations made with G. Godefroy (2011).

2. Tightness

Proposition (F. - Godefroy)

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3. *there is a sequence of subsets $I_0 < I_1 < I_2 < \ldots$ of $\mathbb{N}$, such that the support on (e_n) of any isomorphic copy of Y intersects all but finitely many of the I_j's.*

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$0-1$ topological laws imply that Y is either tight in X, or embeds in a comeager class of block-subspaces of X. But a much more powerful result is true.

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3. *there is a sequence of subsets (intervals) $I_0 < I_1 < I_2 < \ldots$ of $\mathbb{N}$, such that the support on (e_n) of any isomorphic copy of Y intersects all but finitely many of the I_j's.*

If (i)-(ii)-(iii) occurs we say that Y is tight in X.

Definition (F. - Rosendal)

A space X is tight if Y is tight in X for any space Y.

So we may reformulate tightness more explicitely as:

2. Tightness

Proposition

Let X be a space with a basis (e_n). Then the following are equivalent

1. *X is tight.*
2. *any (block-subspace) Y embeds in no more than a meager class of block-subspaces of X (or the equivalent in the Cantor space setting)*
3. *for any (block-subspace) Y, there is a sequence of subsets (intervals) $I_0 < I_1 < I_2 < \ldots$ of $\mathbb{N}$, such that the support on (e_n) of any isomorphic copy of Y intersects all but finitely many of the I_j's.*

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Theorem (3d dichotomy, F. - Rosendal, 2007)

Every Banach space contains a minimal subspace or a tight subspace.

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Before seeing how this may improve Gowers' classification, let us see how special types of tightness may be defined according to the way the I_j's may be chosen in function of Y in 3.

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For example, if Y is a block-subspace $[y_n]_{n\in\mathbb{N}}$ of X, a natural choice is $I_j = \mathrm{supp}\ y_j$ for all j.

Lemma

Let X be a space with a basis. The following are equivalent:

1. *X is tight and for every block subspace $Y = [y_j] \subset X$, the tightness of Y in X is witnessed by the sequence* $I_j = \mathrm{supp}\, y_j$
2. *no subspace of X embeds in X disjointly from its support,*
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2. Forms of tightness

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So we recover Gowers' space G_u's main property. We shall call this property of G_u tightness by support.

Also Gowers' 2nd dichotomy is interpreted as between a strong form of tightness and a weak form of minimality.

In passing, note that Gowers' classification is therefore refined as follows:

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Every Banach space contains a subspace with one of the four properties:

- tight and HI (a subspace of GM),
- tight by support (G_u),
- tight, quasi-minimal with an unconditional basis (T),
- minimal (c_0, ℓ_p, T^*, S).

2. Four classes revisited

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To further divide these classes, we shall now recall the notion of range of a vector.

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If X is a space with a basis $(e_i)_i$, and $x = \sum_{i=0}^{\infty} x_i e_i \in X$, then

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If $Y = [y_n, n \in \mathbb{N}]$ is a block subspace of X, then the support of Y is $\cup_{n \in \mathbb{N}} \mathrm{supp}\ y_n$, and the range of Y is $\cup_{n \in \mathbb{N}} \mathrm{ran}\ y_n$.

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So say $[e_1 + e_2, e_5 + e_6, ...]$ and $[e_3 + e_4, e_7 + e_8, ...]$ have dijsoint ranges,
but $[e_1 + e_3, e_5 + e_7, ...]$ and $[e_2 + e_4, e_6 + e_8, ...]$ have disjoint supports but not disjoint ranges.

2. Tightness by range

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Observe that if (x_n) is an even-odd block-sequence, then $[x_{2n}]$ embeds disjointly from its range. Therefore by 2., tightness by range may be seen as a slightly stronger form of Casazza's criterion.

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Observe that if (x_n) is an even-odd block-sequence, then $[x_{2n}]$ embeds disjointly from its range. Therefore by 2., tightness by range may be seen as a slightly stronger form of Casazza's criterion. The two properties are so similar that we shall give ideas of some proofs in the case of Casazza's criterion instead of tightness by range.

2. Tightness by range

Question

Is tightness by range really weaker than tightness by support?

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Theorem (F. - Rosendal, 07)

Yes. Gowers' asymptotically unconditional and HI *space* G_{au} *is tight by range.*

However it is not tight by support, since it is HI.

We shall now see that there also exists a dichotomy relative to tightness by range.

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- If X is subsequentially minimal, then a subsequence embeds into a very flat, wlog disjointly ranged, block-sequence - therefore X is not tight by range.
- if X is saturated with even-odd block sequences, use Gowers' Ramsey theorem to enumerate, as a block sequence, sufficiently many vectors witnessing the equivalences.

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Unconditional basis	$**$ 1st dichotomy $**$	HI
$\Uparrow$		$\Downarrow$
Tight by support	$**$ 2nd dichotomy $**$	Quasi minimal
$\Downarrow$		$\Uparrow$
Tight by range	$**$ 4th dichotomy $**$	Seq. minimal (*)
$\Downarrow$		$\Uparrow$
Tight	$**$ 3rd dichotomy $**$	Minimal

(*) Sequential minimality is a hereditary version of subsequential minimality.

Theorem (F. - Rosendal 2007)

Any infinite dimensional Banach space contains a subspace of one of the types listed in the following chart:

Type	Properties	Examples
(1)	HI, tight by range	G_{au}
(2)	HI, tight, sequentially minimal	?
(3)	tight by support	G_u
(4)	unconditional basis, tight by range, quasi minimal	?
(5)	unconditional basis, tight, sequentially minimal	T, $T^{(p)}$
(6)	unconditional basis, minimal	S, T^*, c_0, ℓ_p

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Joint work with C. Rosendal, 2007
3. Properties of Gowers and Maurey's spaces
Joint work with Th. Schlumprecht, 2011

3. Type (2) spaces

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So we just needed to "look" at the first known HI space to obtain a type (2) space!

Theorem

Any infinite dimensional Banach space contains a subspace of one of the types listed in the following chart:

Type	Properties	Examples
(1)	HI, tight by range	G_{au}
(2)	HI, tight, sequentially minimal	$\mathcal{X}_{GM}$
(3)	tight by support	G_u
(4)	unconditional basis, tight by range, quasi minimal	?
(5)	unconditional basis, tight, sequentially minimal	T, $T^{(p)}$
(6)	unconditional basis, minimal	S, T^*, c_0, ℓ_p

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3. special functionals built from the x_n^* show that a combination of the x_i's has norm much larger than the corresponding combination of the y_i's, contradicting equivalence.

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So this is why G_u and G_{au} satisfy Casazza's criterion, but the question remained for *GM*.

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- By the HI property, we may find in two disjointly supported subspaces two normalized sequences (x_n) and (y_n), so that $\|x_n - y_n\|$ tends fast enough to 0. Then the map $x_n \mapsto y_n - x_n$ is compact, and therefore x_n is equivalent to y_n, and disjointly supported.

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- So we build $x_1 < y_1 < x_2 < y_2 < \cdots$ so that $x_n \mapsto x_n - y_n$ (and $y_n \mapsto x_n - y_n$) is bounded and strictly singular.

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- this is possible using functionals with multiple weights, thanks to the "yardstick vectors" of Kutzarova - Lin.

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But we conjecture that *GM* itself is saturated with even-odd sequences.

Many interesting questions relative to a different form of tightness (of a more local nature) also remain unsolved. And also of course the existence of a type (4) space.

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