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Introduction to the Gan-Gross-Prasad conjecture

F local or global field, char $F = 0$
E/F quadratic ext^n ($E = F \times F$ allowed)
$(V, \langle\,\rangle)$ Hermitian space $/E$ of dim$_E$ n, nondegenerate
$U(V)$ = corresponding unitary group.
$(E = F \times F \Rightarrow U(V) = GL_n(F))$
$W \subset V$ nondegen. codim 1 subspace
$G := U(V) \times U(W)$ $\quad (V = W \oplus W^{\perp})$
$H := U(W)$
diagonal embedding $H \hookrightarrow G$.

Main question: (F global)

Π, Σ irred admissible reps of $U(V)(\mathbb{A}_F)$, $U(W)(\mathbb{A}_F)$
$\Pi \boxtimes \Sigma$ irred rep. of $G(\mathbb{A}_F)$.
Ask:
Is $\mathrm{Hom}_{H(\mathbb{A}_F)}(\Pi \boxtimes \Sigma, \mathbb{C})$ nonzero?

Global motivation:
Π, Σ cuspidal automorphic reps of $U(V), U(W)$

$$P : \Pi \times \Sigma \longrightarrow \mathbb{C}$$
$$f, \varphi \longmapsto \int_{H(F)\backslash H(\mathbb{A}_F)} f(h)\varphi(h)\,dh.$$

Ask: Is this nonzero?

Notice: $P \in \mathrm{Hom}_{H(\mathbb{A}_F)}(\Pi \boxtimes \Sigma, \mathbb{C})$

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So P can only be nonzero only if $\mathrm{Hom}_{H(\mathbb{A}_F)}(\Pi \boxtimes \Sigma, \mathbb{C})$ is nonzero.

$$\Pi \otimes \Sigma$$
$$\overset{\shortparallel}{\otimes \Pi_v} \qquad \overset{\shortparallel}{\otimes \Sigma_v}$$

$\downarrow$ F now local, $\Pi \sim \Pi_v$, $\Sigma \sim \Sigma_v$

<u>Thm</u> (Multiplicity one result) (Aizenbud, Gurevich, Rallis, Schiffmann p-adic case)

$$\dim \mathrm{Hom}_{H(F)}(\Pi \boxtimes \Sigma, \mathbb{C}) \leq 1$$

(B. Y. Sun, C. B. Zhu archimedean case)

<u>Rmk.</u> If $E = F \times F$, $G = GL_n \times GL_{n-1}$, then

$$\mathrm{Hom}_{GL_{n-1}(F)}(\Pi \boxtimes \Sigma, \mathbb{C}) \neq 0$$

when Π, Σ are generic (given by local Rankin-Selberg integrals) (in particular, when Π, Σ are local components of global cuspidal auto. reps)

<u>Local GGP (unitary case)</u>

- conjecture in terms of local Langlands correspondence for unitary groups & their pure inner forms.

<u>Local Langlands corresp. for GL_n</u> (Harris-Taylor, Henniart)

$$\left\{ \begin{array}{l} \text{irred admissible} \\ \text{reps of } GL_n(F) \end{array} \right\} \qquad \left\{ \begin{array}{l} \text{admissible homomorphisms} \\ \varphi : WD_F \to GL_n(\mathbb{C}) \end{array} \right\}$$

$$\overset{\shortparallel}{\mathrm{Irr}\, GL_n(F)} \qquad\qquad \overset{\shortparallel}{\Phi(GL_n(F))}$$

s.t. ....

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encodes monodromy

$$WD_F := W_F \times \underbrace{SL_2(\mathbb{C})}$$

For $G \neq GL_n$, this is not necessarily true.

<u>Notation</u>: ${}^LG = \hat{G} \rtimes W_F$

For our purposes, we need:

- inner forms have the same L-groups.

In particular, we may take G to be quasisplit in defining the L-group.

<u>Pure inner forms</u>

- inner forms classified by $H^1(F, G/Z)$
- pure inner forms: $H^1(F, G)$

In our situation, $G = U(V)$, $\hat{G} = GL_n$

$${}^LG = GL_n(\mathbb{C}) \rtimes Gal\, E/F = \langle \sigma \rangle$$

$\langle \sigma \rangle \circlearrowright$

$\sigma g := \phi_n {}^t g^{-1} \phi_n^{-1}$, $\phi_n = \begin{pmatrix} & & 1 \\ & (-1) & \\ (-1)^{n-1} & & \end{pmatrix}$ ← quasisplit

<u>Conjectural LLC for G_0 & its pure inner forms (Vogan)</u> ${}^t\phi_n = (-1)^{n-1}\phi_n$.

$\exists$ surjection with finite fibres

$$\coprod_{\substack{G \text{ pure inner} \\ \text{form of } G_0}} \operatorname{Irr} G(F) \longrightarrow \Phi(G_0(F))$$

L-parameters.

~~fibres = L-packets~~

If $G_0 = U(V_0)$, then $G = U(V)$ where $\dim V = \dim V_0$.

Given $\phi \in \Phi(G_0(F))$, its inverse image Π_ϕ under the above is called the Vogan Packet

Conjectural parametrization of elements in Vogan packet:

$$\Pi_\phi \xrightarrow{\sim} \hat{A}_\phi \quad \text{where } A_\phi = \pi_0(\mathrm{Cent}_G \operatorname{im}\phi)$$

depends on choice of Whittaker data

G unitary group $\Rightarrow A_\phi$ = finite abelian 2-group.

Furthermore, for $U(V) = G$, an L-param. $\varphi \in \Phi(G)$ can be described as a homomorphism

$$\phi : WD_E \longrightarrow GL_n(\mathbb{C})$$

that is conjugate self-dual such that φ is

$$\begin{cases} \text{conjugate-symplectic} & \text{if } n \text{ is even} \\ \text{conjugate-orthogonal} & \text{if } n \text{ is odd.} \end{cases}$$

So if $\pi \in \Pi_\phi$, then π can be parametrized as $\pi(\varphi, \chi)$, $\chi \in \hat{A}_\varphi$. The pure inner form is determined as follows:

G_0 = quasi-split inner from

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Thm. (Kottwitz) Nakayama-Tate duality (F p-adic)

$$H^1(F, G) = \pi_0\left(\underset{\substack{\parallel \\ Z({}^L G_0)}}{Z(\hat{G}_0)^{\mathrm{Gal}\,\bar{F}/F}_{\text{"}\Gamma_F\text{"}}}\right)^{\text{dual}}$$

So given $\phi \in \Phi(G_0(F))$ & $\chi \in \hat{A}_\varphi$,

$$Z(\hat{G})^{\Gamma_F} \hookrightarrow \mathrm{Cent}_{\hat{G}}\, \mathrm{im}\,\varphi.$$

take π_0

$$\pi_0\left(Z(\hat{G})^{\Gamma_F}\right) \longrightarrow A_\varphi = \pi_0\left(\mathrm{Cent}_{\hat{G}}\, \mathrm{im}\,\varphi\right)$$

Arthur treats Sp_{2n}, SO_n & claims the unitary case can be treated similarly.

In the local GGP conjecture, the characters $\chi \in A_\varphi$ are given by local symplectic root numbers.

First, suppose φ is an L-param. for $U(V)$. Identify

$$\phi = m_1\phi_1 \oplus \cdots \oplus m_r\phi_r, \quad \underset{\text{irred.}}{\phi_i : WD_E} \to GL_{n_i}(\mathbb{C})$$

Even if ϕ is conjugate self dual, the ϕ_i need not be.

In this case, A_φ can be described easily in terms of this data:

$$A_\varphi = \bigoplus_{i \in S} \mathbb{Z}/2\mathbb{Z}\, a_i, \text{ where}$$

S = set of indices st φ_i is conjugate self dual of same parity as φ.

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Given L-parameters φ & $\tilde{\varphi}$, $\varphi \in \Phi(U(\overset{\text{dim } n}{V_0}))$, $\tilde{\varphi} \in \Phi(U(\overset{\text{dim } n-1}{W_0}))$

A relevant pure inner form of $G_0 = U(V_0) \times U(W_0)$ (quasisplit) of G_0 is one s.t

$$W \subset V \ \& \ V/W \cong V_0/W_0.$$

<u>Conjecture</u> (local GGP, special case)

$\varphi = m_1 \varphi_1 + \cdots + m_r \varphi_r$, $\tilde{\varphi} = n_1 \tilde{\varphi}_1 + \cdots n_t \tilde{\varphi}_t$

$A_\varphi = \bigoplus_{i \in S} \mathbb{Z}/2\, a_i$, $\quad A_{\tilde{\varphi}} = \bigoplus_{k \in \tilde{S}} \mathbb{Z}/2\mathbb{Z}\, b_k$.

The unique repn $\pi \otimes \sigma$ in the Vogan packet ~~the~~ $\Pi_\varphi \times \Pi_{\tilde{\varphi}}$ having ~~...~~ s.t

$$\mathrm{Hom}_{H(F)}(\pi \boxtimes \sigma, \mathbb{C}) \neq 0 \qquad \left(\begin{array}{l}\text{recall } G = U(V) \times U(W), \\ H = U(W)\end{array}\right)$$

is indexed by the character $\chi \times \tilde{\chi} \in \hat{A}_\varphi \times \hat{A}_{\tilde{\varphi}}$, where

$$\chi(a_i) = \varepsilon(\tfrac{1}{2}, \varphi_i \otimes \tilde{\varphi}, \psi) = \pm 1,$$

& similarly for $\tilde{\chi}(b_k)$. Here $\psi : E/F \to S^1$ is a nontrivial additive character.

<u>Remark</u>: Can read off the relevant pure inner form $U(V) \times U(W)$ can be read off of the values of

$$\chi(\text{image of } (-I_n) \text{ in } A_\varphi) = \varepsilon(\tfrac{1}{2}, \varphi \otimes \tilde{\varphi}, \psi)$$

recall: $H^1(F, G_0) \cong \pi_0(Z(\hat{G}_0)^{\Gamma_F})^{\text{dual}}$

$G_0 = U(V_0)$

$Z(\hat{G}_0)^{\Gamma_F} = \{\pm I_n\}$