Trivial zeros of p-adic L-functions at near central points

(Banff, 01/11/2011)

① Kubota-Leopoldt L-function

$\eta : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ primitive Dirichlet character

$\eta(-1) = -1$

$p \nmid N$ fixed odd prime

$\omega : (\mathbb{Z}/p\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ Teichmüller character

$$L_p(\eta\omega, 1-j) = (1-(\eta\omega^{1-j})(p)\, p^{j-1})\, L(\eta\omega^{1-j}, 1-j)$$

$j \geq 1$

(interpolation property)

- $L(\eta, 0) \neq 0$
- $L_p(\eta\omega, 0) = 0 \iff \eta(p) = 1$ (trivial zero)

$$L_p'(\eta\omega, 0) = -\mathcal{L}(\eta)\, L(\eta, 0)$$ Ferrero-Greenberg + Gross-Koblitz (1979)

Dasgupta-Darmon-Pollack (2009):
direct proof using families + conditional generalization to totally real fields

Definition of $\mathcal{L}(\eta)$:

$$\mathrm{Im}(H^1(\mathbb{Q}, L(\eta)) \to H^1(\mathbb{Q}_p, L)) = L(\log\chi + \mathcal{L}(\eta)\,\mathrm{ord}_p)$$

$$H^1(\mathbb{Q}_p, L) = L(\log\chi) + L(\mathrm{ord}_p)$$

cyclotomic character

$\mathrm{ord}_p : \mathrm{Gal}(\mathbb{Q}_p^{nr}/\mathbb{Q}_p) \to \mathbb{Z}_p$

(2) $f \in S_k(\Gamma_0(N), \varepsilon)$, $k \geq 2$

$$\| \quad \sum_{n\geq 1} a_n q^n \leftarrow \text{normalized primitive.}$$

α = root of $X^2 - a_p X + \varepsilon(p) p^{k-1}$: $v_p(\alpha) < k-1$

$$L_{p,\alpha}(f, \omega^m, j) = \underbrace{\mathcal{E}_\alpha(f, \omega^m, j)}_{\text{Euler-like factor}} \tilde{L}(f, \omega^{j-m}, j)$$

$$\tilde{L}(f, \omega^{j-m}, j) = \frac{\Gamma(j) L(f, \omega^{j-m}, j)}{(2\pi i)^{j-1} \Omega_f^{\pm}}$$

$1 \leq j \leq k-1$

Def $L_{p\alpha}$ has an extra zero at $s=j \Leftrightarrow \mathcal{E}_\alpha(f, \omega^m, j) = 0$

——— " ——— trivial zero $\Leftrightarrow$ extra zero $+ \tilde{L}(f, \omega^{j-m}, j) \neq 0$

$$\mathcal{E}_\alpha(f, \omega^m, j) \begin{cases} \neq 0, & j \not\equiv m \,(\mathrm{mod}(p-1)) \\ = \left(1 - \frac{p^{j-1}}{\alpha}\right)\left(1 - \frac{\varepsilon(p) p^{k-j-1}}{\alpha}\right), & j \equiv m \,(p-1) \end{cases}$$

Phenomenon first studied by Mazur-Tate-Teitelbaum (1986).

$\rho_f : G_{\mathbb{Q}} \longrightarrow GL(W_f)$ Deligne representation

W_f = 2-dimensional over $L/\mathbb{Q}_p$.

<u>Semistable non-crystalline case</u> : $p \| N$, $(p, \mathrm{cond}(\varepsilon)) = 1$

($\Rightarrow \rho_f$ is semi-stable, non-crystalline)

$\mathcal{E}_\alpha(f, \omega^m, j) = 0 \Leftrightarrow$
- k even
- $j = k/2$
- $a_p = p^{k/2-1}$
- $m \equiv j \,(\mathrm{mod}(p-1))$

<u>MTT conjecture</u> $L'_{p,\alpha}(f,\omega^{k/2},k/2) = \mathcal{L}_{FM}(f)\tilde{L}(f,k/2)$.

Fontaine-Mazur $\mathcal{L}$-invariant.

Proofs: • by Euler systems (Kato-Kurihara-Tsuji) $\approx$ 1996-2000

• using Hida theory and Coleman theory (Greenberg-Stevens, Stevens) $\approx$ 1994-2000

Definition of $\mathcal{L}_{FM}(f)$:

$$\mathbb{D}_{st}(W_f) = L e_\alpha + L e_\beta; \quad N e_\beta = e_\alpha$$

$$\varphi(e_\alpha) = a_p e_\alpha, \ \varphi(e_\beta) = p a_p e_\beta$$

$$\cup$$

$$\mathrm{Fil}^{k-1}\mathbb{D}_{st}(W_f) = L(e_\beta - \mathcal{L}_{FM}(f)\, e_\alpha)$$

General definitions of $\mathcal{L}$-invariant

• Greenberg (1994) $M/\mathbb{Q}$ motive ordinary at p

• B. (2008) $M/\mathbb{Q}$ semistable at p — more general

Main goal of this talk is to apply the construction from B. (2008) to:

③ <u>Crystalline case</u> $p \nmid N$ ($\Rightarrow W_f$ is crystalline)

$\mathcal{E}_\alpha(f,\omega^m,j) = 0 \iff$ • k odd

• either

near central points

$j = \frac{k+1}{2}$, $\alpha = p^{(k-1)/2}$ or

$j = \frac{k-1}{2}$, $\alpha = \varepsilon(p)\, p^{(k-1)/2}$

$m \equiv j \pmod{p-1}$ in the both cases.

Th (B.) Let $\alpha = p^{\frac{k-1}{2}}$. Assume that φ acts semisimply on $\mathbb{D}_{cris}(W_f)$.
Then

$$L_{p,\alpha}\left(f, \omega^{\frac{k+1}{2}}, \frac{k+1}{2}\right) = -\mathcal{L}_\alpha(f)\left(1-\frac{\varepsilon(p)}{p}\right)\tilde{L}\left(f,\frac{k+1}{2}\right)$$

our $\mathcal{L}$-inv.

Remarks: ① $\tilde{L}\left(f, \frac{k+1}{2}\right) \neq 0$ (Jacquet - Shalika)

②

Kubota-Leopoldt	near central	$\mathcal{L}$ is global
semistable $\neq$ crys	central point	$\mathcal{L}$ local
crystalline	near central points	$\mathcal{L}$ global.

We will show that

$$\left.\begin{array}{l}\text{Euler systems +}\\ \text{some result about}\\ \text{Perrin-Riou exp. map}\end{array}\right\} \Longrightarrow ①, ②, ③$$

(φ, Γ)-modules $\quad \Gamma = \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \overset{\chi}{\simeq} \mathbb{Z}_p^\times$

$L/\mathbb{Q}_p$ finite (field of coefficients)

$$\mathbb{D}^{+}_{rig} : \left\{\begin{array}{l}L\text{-adic}\\ \text{representations}\\ \text{of } G_{\mathbb{Q}_p}\end{array}\right\} \xrightarrow{\text{fully faithful}} \left\{\begin{array}{l}(\varphi,\Gamma)\text{-modules}\\ \text{over } \mathcal{R}_L\end{array}\right\}$$

(Fontaine, Cherbonnier - Colmez, Kedlaya)

Example $\quad \delta : \mathbb{Q}_p^\times \to L^\times$ continue

$\mathcal{R}_L(\delta) := \mathcal{R}_L e_\delta \qquad \varphi(e_\delta) = \delta(p)\, e_\delta$

$\qquad \gamma(e_\delta) = \delta(\chi(\gamma))\, e_\delta$

$\mathcal{L}$-invariant of modular forms

$$k_0 := \begin{cases} k/2 & \text{semistable case} \\ (k+1)/2 & \text{crystalline case.} \end{cases}$$

$$V_f := W_f(k_0)$$

Assume $\alpha = p^{k_0-1}$ (trivial zero)

$$\begin{tikzcd} D_\alpha = \mathbb{D}_{st}(V_f)^{\varphi=p^{-1}} \arrow[r, phantom, "\subset"] \arrow[d, leftrightarrow, "\text{Berger theory}"'] & \mathbb{D}_{st}(V_f) \arrow[d, leftrightarrow] \\ \mathcal{R}(\delta) \arrow[r, phantom, "\subset"] & \mathbb{D}^{+}_{rig}(V_f) \end{tikzcd}$$

$\delta(x) = |x| x^{k_0}$

$$0 \to \mathcal{R}_L(\delta) \to \mathbb{D}^{+}_{rig}(V_f) \to \mathcal{R}_L(\delta') \to 0$$

(triangulation of V, see Colmez, also Bellaiche-Chenevier, Pottharst for other applications)

Definition in the crystalline case

- $H^1(\mathcal{R}_L(\delta)) \simeq H^1(\mathbb{D}^{+}_{rig}(V_f)) \simeq H^1(\mathbb{Q}_p, V_f)$

Because $\delta = |x| x^{k_0}$ — general fact (Herr, Liu)

- $H^1(\mathcal{R}_L(\delta)) \simeq H^1_f(\mathcal{R}_L(\delta)) \oplus H^1_c(\mathcal{R}_L(\delta))$

$H^1_f(\mathcal{R}_L(\delta)) \simeq D_\alpha$, canonically $H^1_c(\mathcal{R}_L(\delta)) \simeq D_\alpha$

- $H^1_f(V_f) = H^1_f(V_f^*(1)) = 0$ (Kato)

Define
$$H^1_{f,\{p\}}(V_f) = \ker\left(H^1(\mathbb{Q}, V_f) \to \bigoplus_{\ell \neq p} \frac{H^1(\mathbb{Q}_\ell, V_f)}{H^1_f(\mathbb{Q}_\ell, V_f)}\right)$$

Poitou-Tate
$$H^1_{f,\{p\}}(V_f) \simeq \frac{H^1(\mathbb{Q}_p, V_f)}{H^1_f(\mathbb{Q}_p, V_f)}$$

$$\begin{array}{ccc} D_\alpha & \simeq & H^1_f(\mathcal{R}_L(\delta)) \\ \uparrow p_f & & \uparrow pr_f \\ H^1_{f,\{p\}}(V_f) & \hookrightarrow & H^1(\mathcal{R}_L(\delta)) \\ \downarrow p_c & & \downarrow pr_c \\ D_\alpha & \simeq & H^1_c(\mathcal{R}_L(\delta)) \end{array}$$

Déf of $\mathcal{L}(f)$:

$$p_f p_c^{-1}(x) = \mathcal{L}_\alpha(f) x$$

$$D_\alpha \to D_\alpha$$

Definition in the general case

V_f is semistable (eventually crystalline)

If V_f semistable non-crystalline we assume $H^1_f(V_f)=0$

(not restrictive because $L(f, k/2) \neq 0 \Rightarrow H^1_f(V_f)=0$)

Dualizing we have $H^1(\mathbb{Q}_p, V_f^*(1)) \to H^1(\mathcal{R}_L(\delta^{-1}\chi))$

and

$$\begin{array}{ccc} D_\alpha^* & \simeq & H^1_f(\mathcal{R}_L(\delta^{-1}\chi)) \\ \uparrow p_f^* & & \uparrow \\ H^1_{f,\{p\}}(V_f^*(1)) & \xhookrightarrow{\varkappa} & H^1(\mathcal{R}_L(\delta^{-1}\chi)) \\ \downarrow p_c^* & & \downarrow \\ D_\alpha^* & \simeq & H^1_c(\mathcal{R}_L(\delta^{-1}\chi)) \end{array}$$

$$\ell_\alpha(f) = \det(p_f^*(p_c^*)^{-1} \mid D_\alpha) \;;\; p_f^*(p_c^*)^{-1}(x) = \ell(f)x.$$

Some homological algebra

$$\ell_\alpha(f) = \begin{cases} -\mathcal{L}_\alpha(f), & \text{crystalline case} \\ \mathcal{L}_{FM}(f) & \text{semistable case.} \end{cases}$$

In the semistable case

$$\operatorname{Im}\varkappa = \operatorname{Im}\left(H^1(\mathbb{Q}_p, V_f^*(1)) \to H^1(\mathcal{R}_L(\delta^{-1}\chi))\right)$$

$\Rightarrow$ $\mathcal{L}$-invariant is local.

Proof of ①-③

$H^1_{Iw}(V_f)$, $H^1_{Iw}(\mathbb{Q}_p, V_f)$ Iwasawa cohomology.

$$\exp^*_{V_f^*(1)} : H^1(\mathbb{Q}_p, V_f^*(1)) \to \mathrm{Fil}^0 \mathbb{D}_{st}(V_f^*(1)) \simeq L$$

dual exponential

fixing compatible bases

$$\mathrm{Log}_{V_f^*(1),\alpha} : H^1_{Iw}(\mathbb{Q}_p, V_f^*(1)) \longrightarrow \mathcal{H}(\Gamma)$$

Perrin-Riou large logarithm.

$z_{Kato} = (z_n)_{n \geq 0} \in H^1_{Iw}(V_f^*(1))$ Kato Euler system.

A) $\exp^*_{V_f^*(1)}(z_0) = E_p(f, p^{-k_0}) \tilde{L}(f, k_0)$

Euler factor

B) $L_{p,\alpha}(f, \omega^{k_0}, s) = \mathrm{Log}_{V_f^*(1)}(z_{Kato})(\chi(\gamma_0)^s - 1)$

$\langle \gamma_0 \rangle = \Gamma$

A) and B) are proved by Kato.

Prop $x = (x_n)_{n \geq 0} \in H^1_{Iw}(V_f^*(1))$ such that $x_0 \neq 0$.

Define $$L_p(\mu_x, s) = \mathrm{Log}_{V_f^*(1),\alpha}(x)(\chi(\gamma_0)^s - 1)$$

Then
- $L_p(\mu_x, k_0) = 0$
- $L_p'(\mu_x, k_0) = \ell_\alpha(f)\left(1 - \frac{1}{p}\right)^{-1} \exp^*_{V_f^*(1)}(x_0)$ ■

A) + B) + Prop $\Longrightarrow$ semistable and crystalline cases.

The case of Kubota-Leopoldt is analogeous.