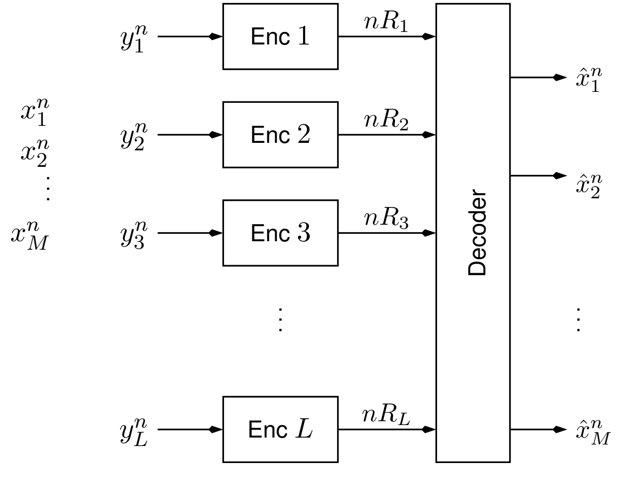
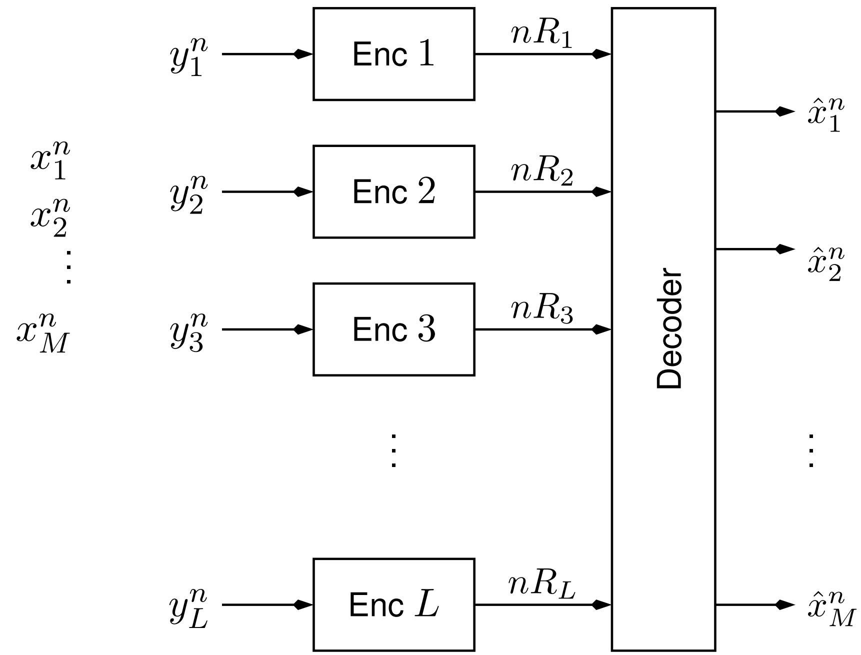
When do Structured Codes Help in Distributed Compression?



Aaron Wagner

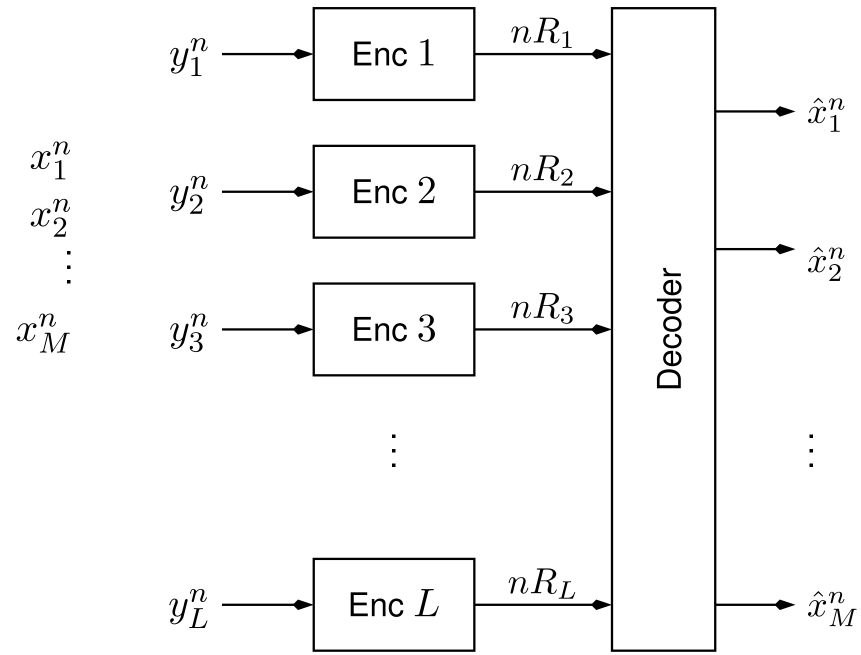
Cornell University

The Problem



$y_1, \ldots, y_L, x_1, \ldots x_M$: jointly Gaussian scalars

The Problem

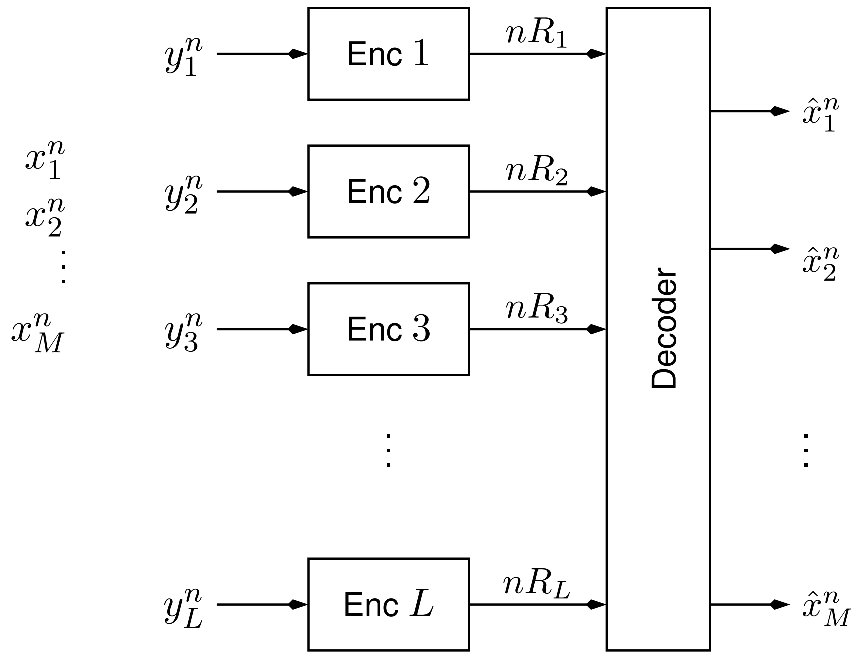


$y_1, \ldots, y_L, x_1, \ldots x_M$: jointly Gaussian scalars

Quadratic Distortion Constraint:

$$\frac{1}{n}\sum_{i=1}^{n} E[(x_\ell(i) - \hat{x}_\ell(i))^2] \leq d_\ell \quad \text{for all } \ell$$

The Problem



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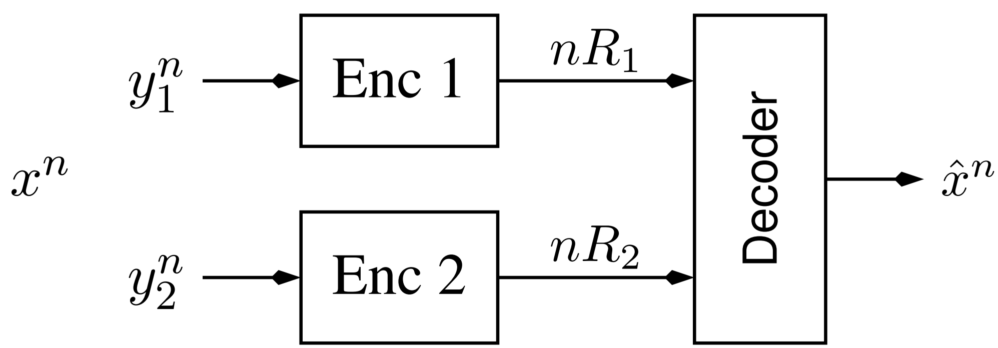
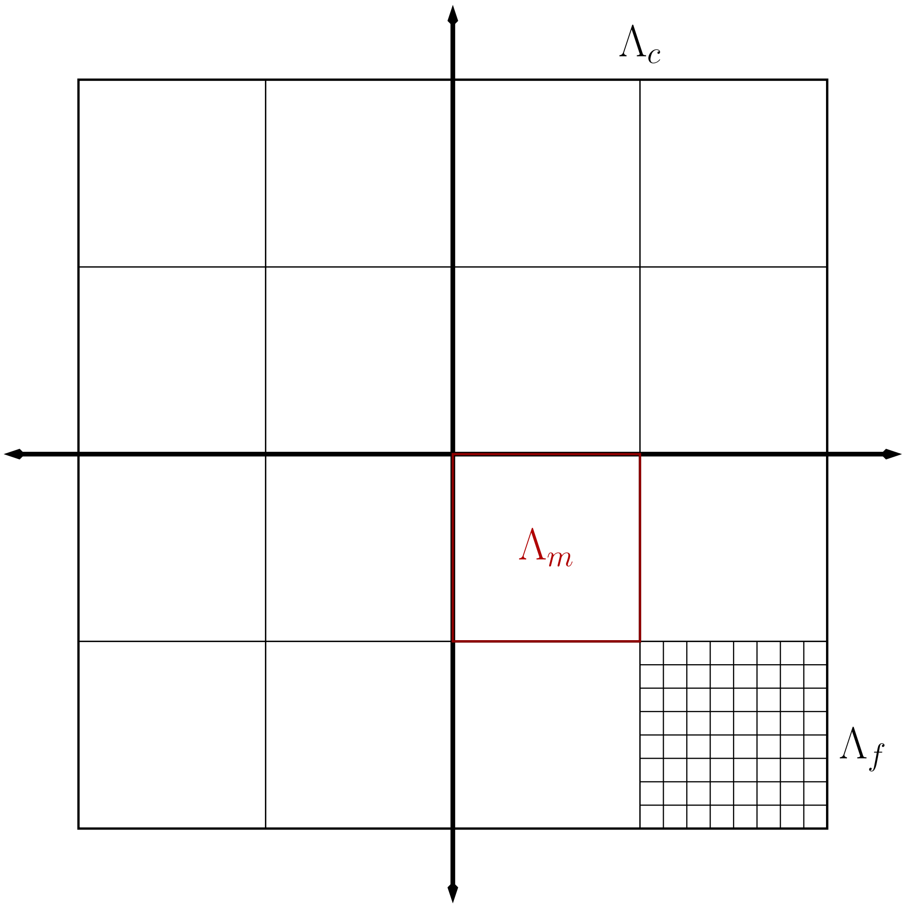
Rate Region?

The Problem



“Unstructured” Scheme



- Quantize y_1^n in Λ_f to $\tilde{y}_1^n$
- Quantize y_2^n in Λ_f to $\tilde{y}_2^n$
- Send $\tilde{y}_1^n \mod \Lambda_c$
- Send $\tilde{y}_2^n \mod \Lambda_m$
- Decoder recovers $\tilde{y}_1^n, \tilde{y}_2^n$
- MMSE

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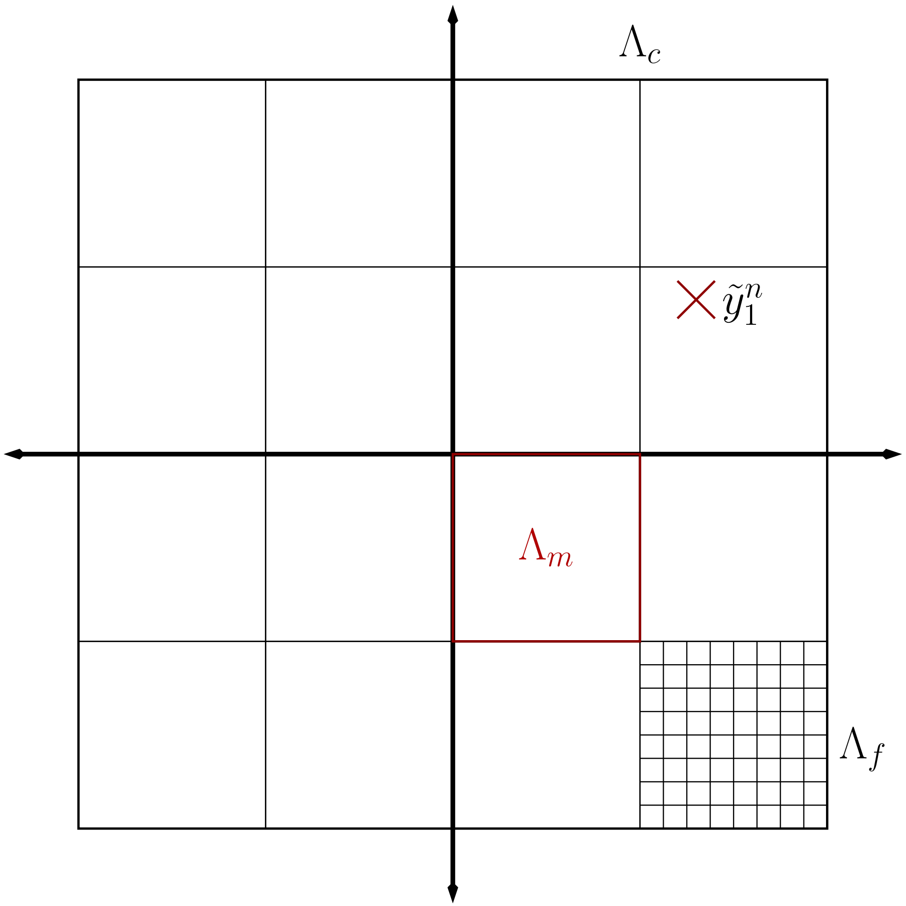

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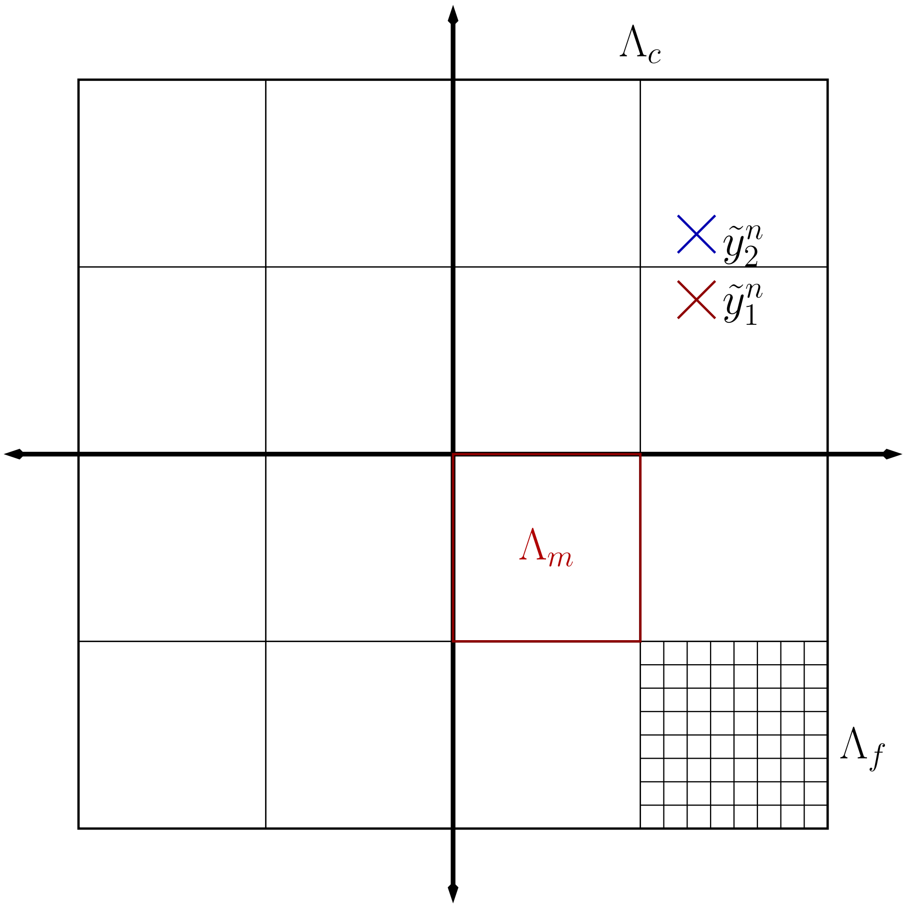

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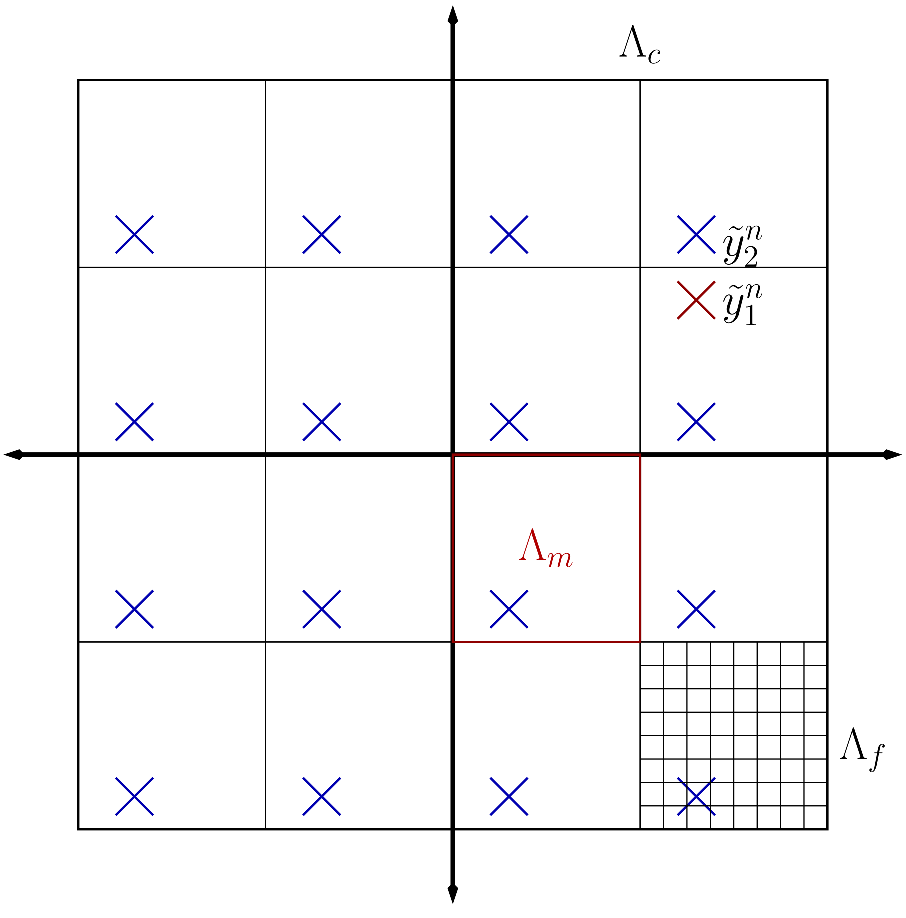

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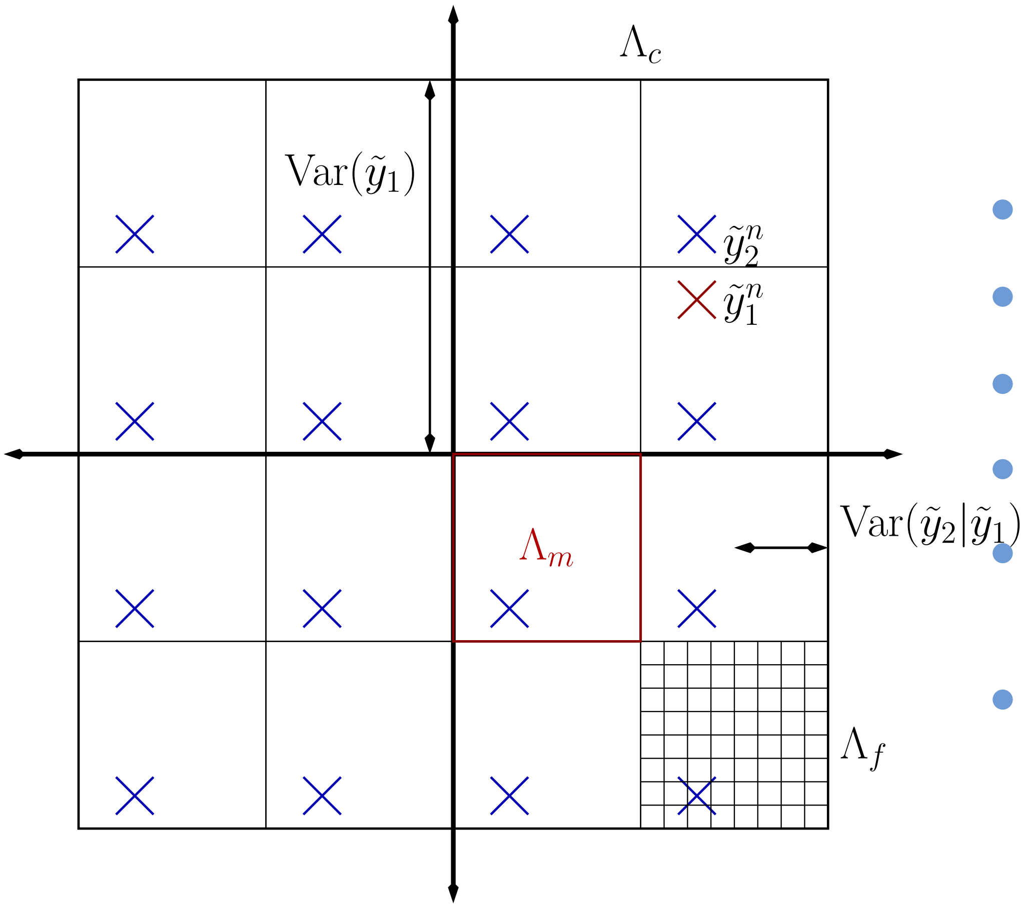

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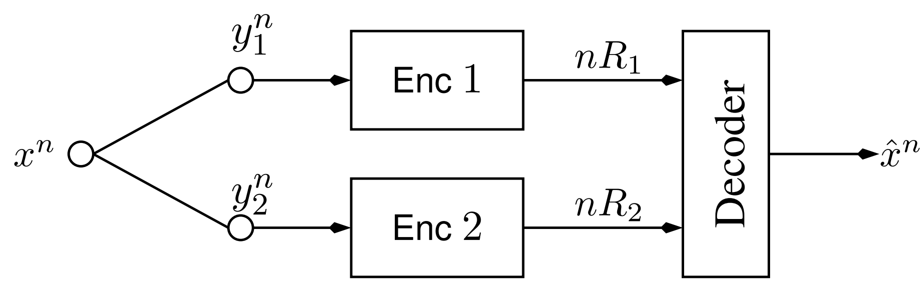
CEO Problem

[formulated by Viswanathan and Berger ’97]



Theorem (Oohama ’99/’05; Prabhakaran, Tse, and Ramchandran ’04): Unstructured scheme is rate-region optimal.

- Key: How much information must inevitably be transmitted about the observation noise?

CEO Reduction

$$\boldsymbol{K}_y = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \quad \rho > 0$$

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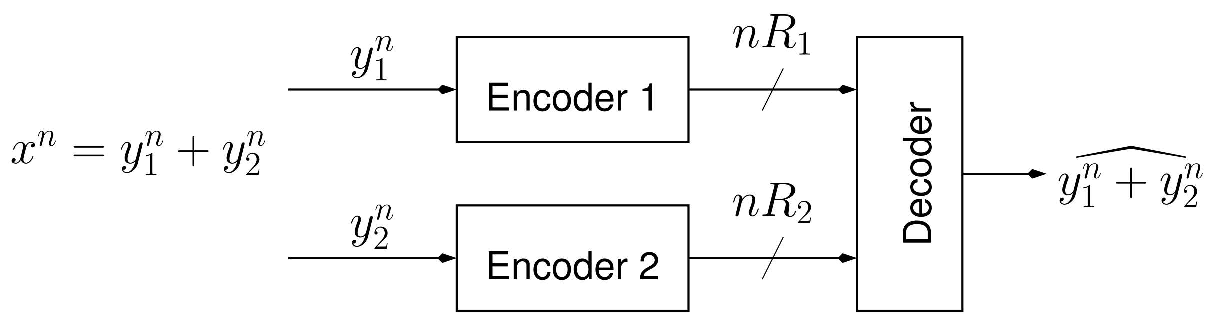

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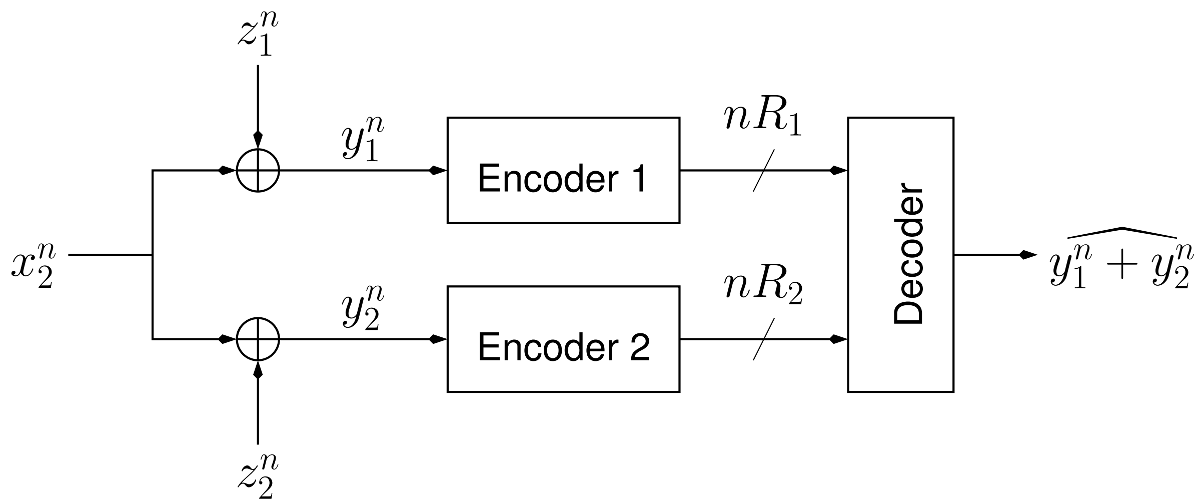

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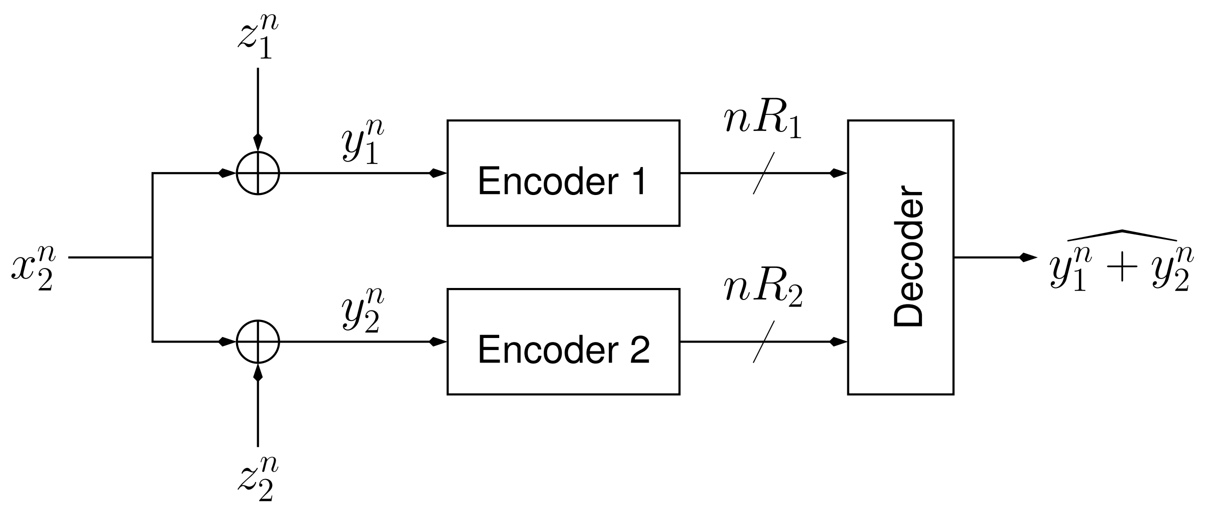
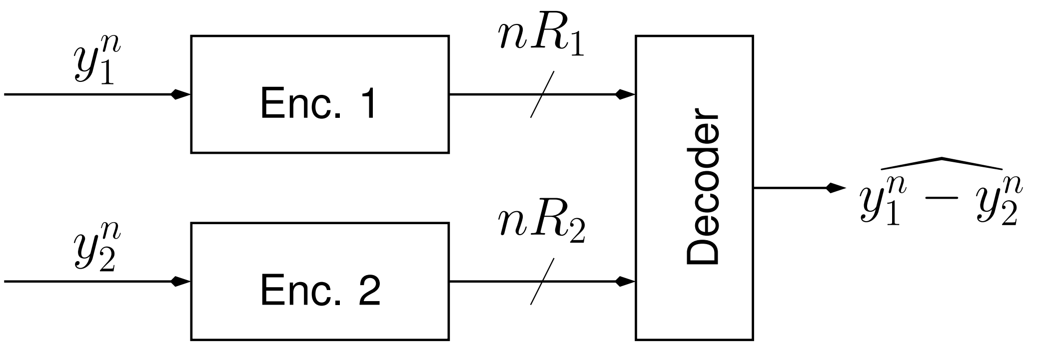


- Unstructured scheme is optimal
- Works for any positive linear combination
- Equivalence class of x variables

$$x_1 \equiv x_2 \text{ if } E[x_1|y_1, y_2] = E[x_2|y_1, y_2]$$

Unstructured Scheme is Not Always Optimal

$$\boldsymbol{K}_y = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \quad \rho > 0$$



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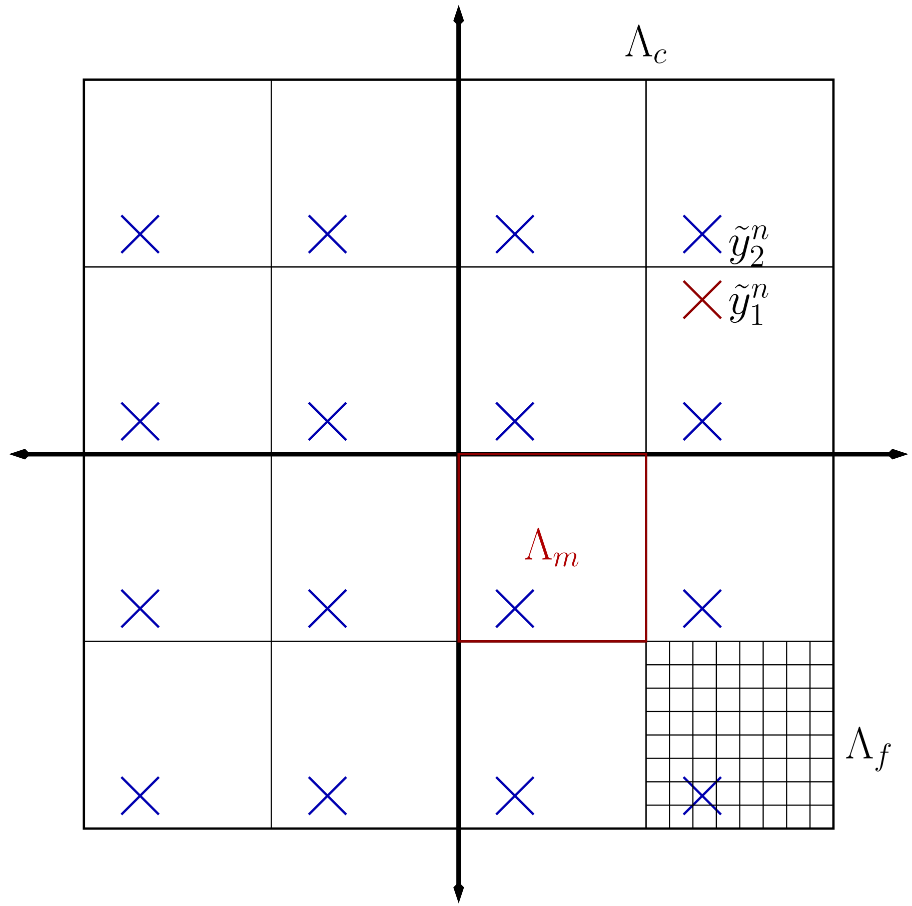

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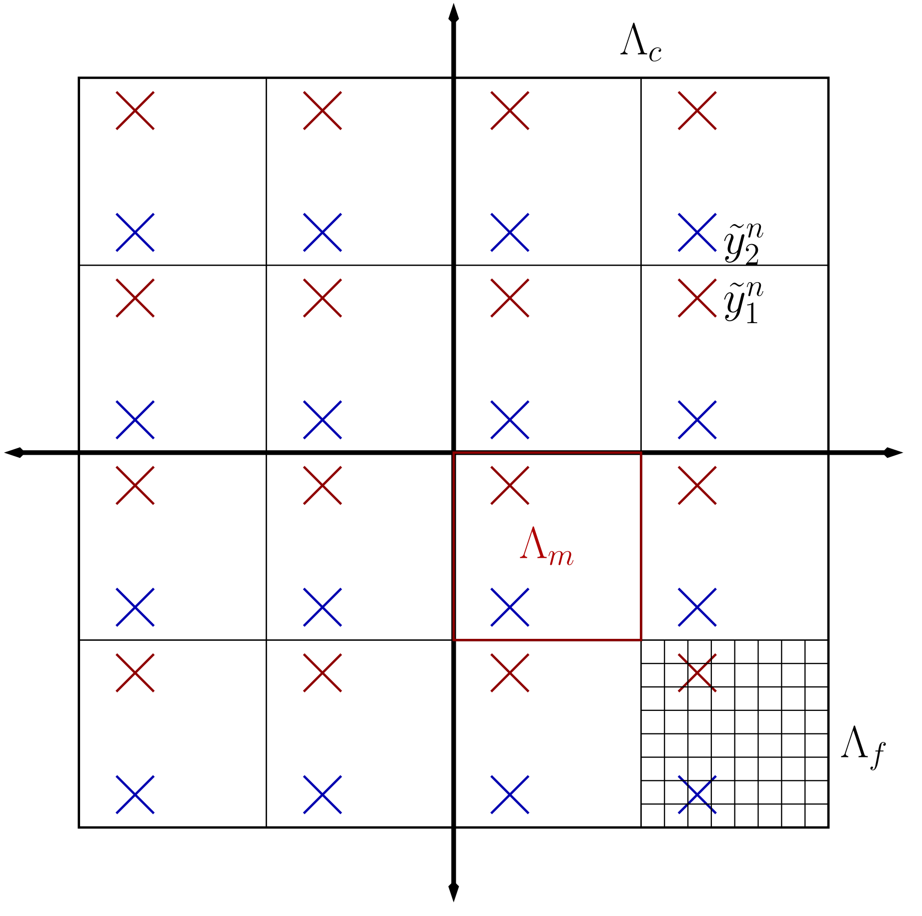

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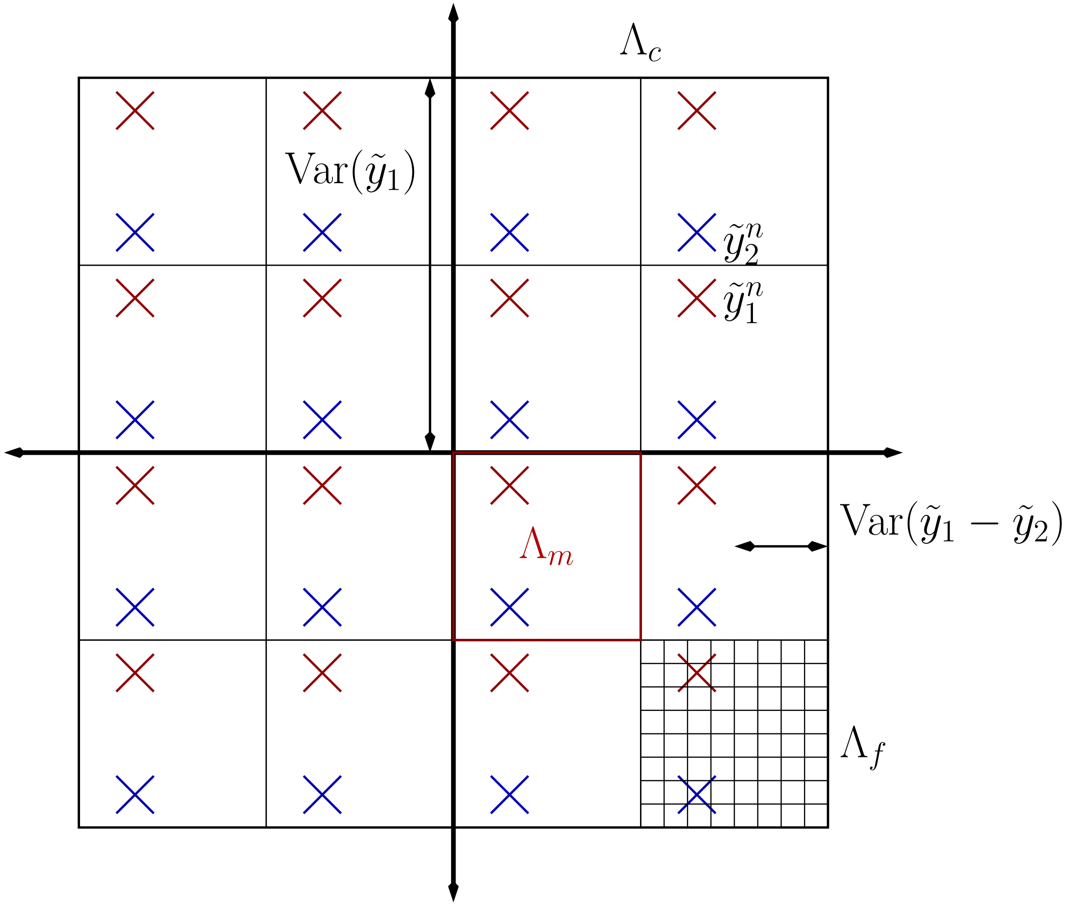

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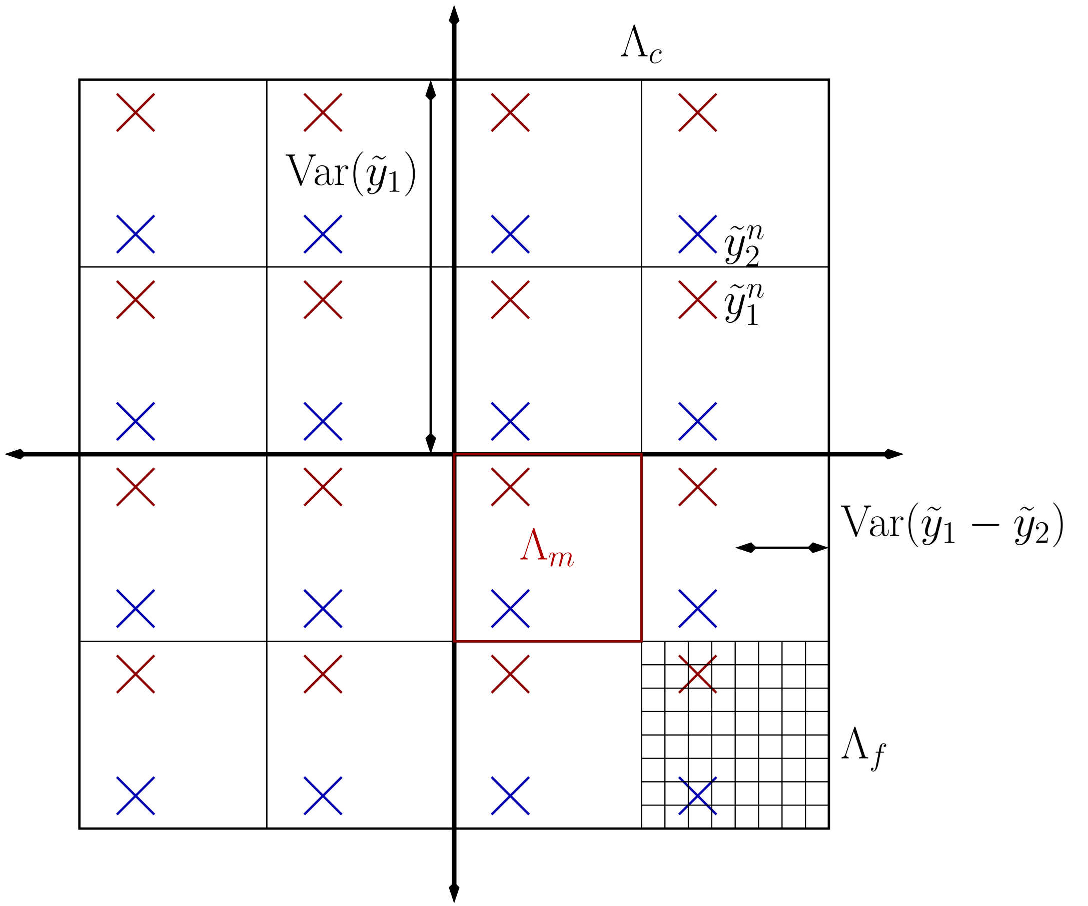
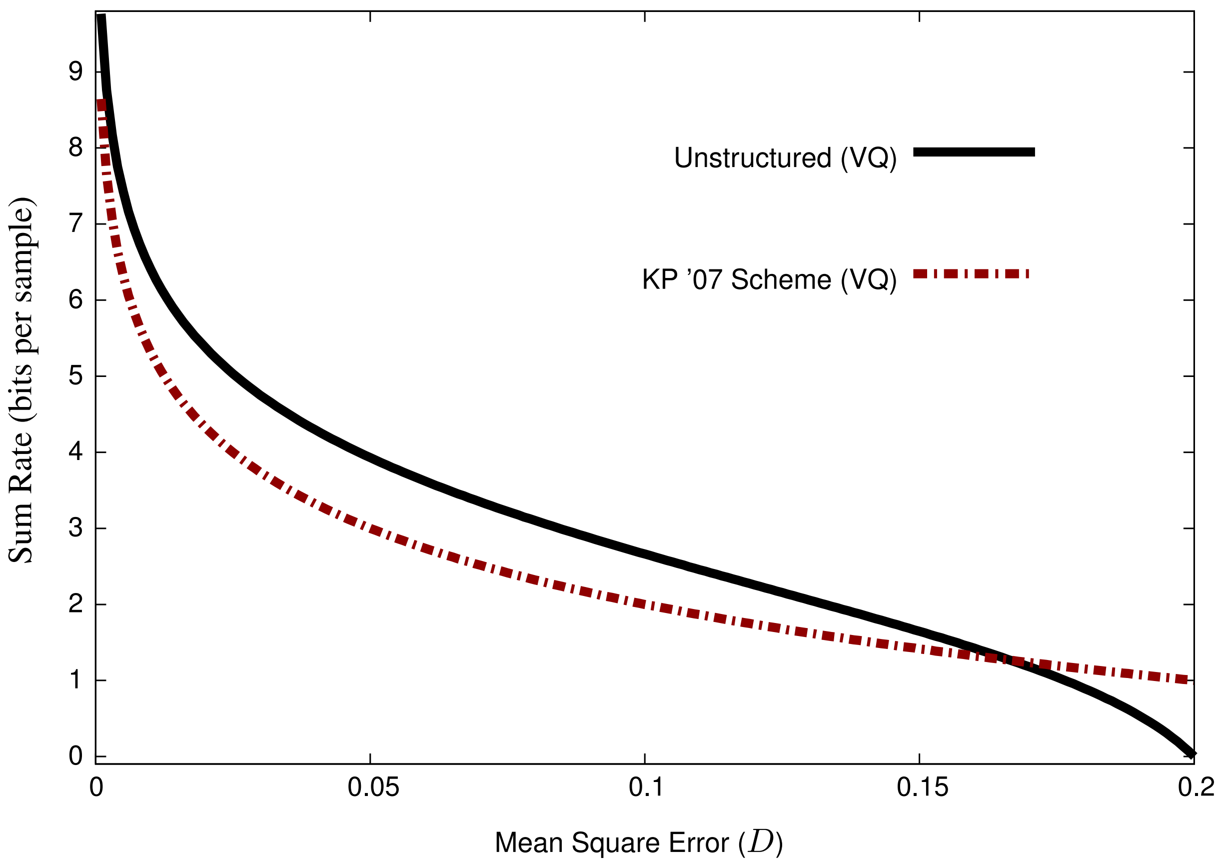


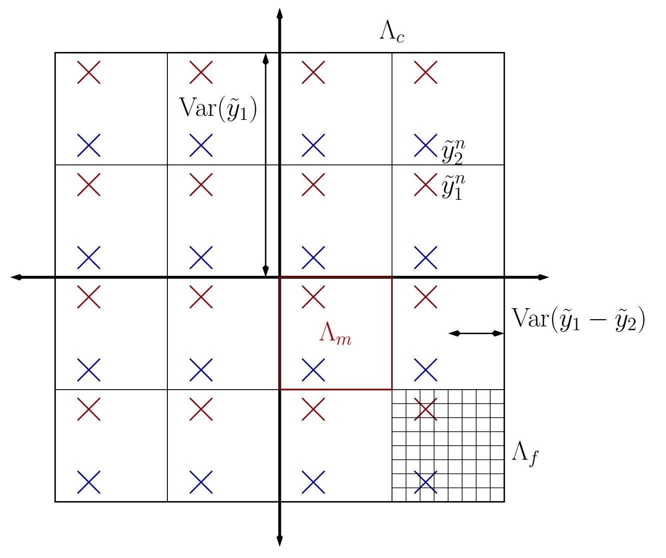
- Körner and Marton ’79
- Krithivasan and Pradhan ’07

Structured vs. Unstructured



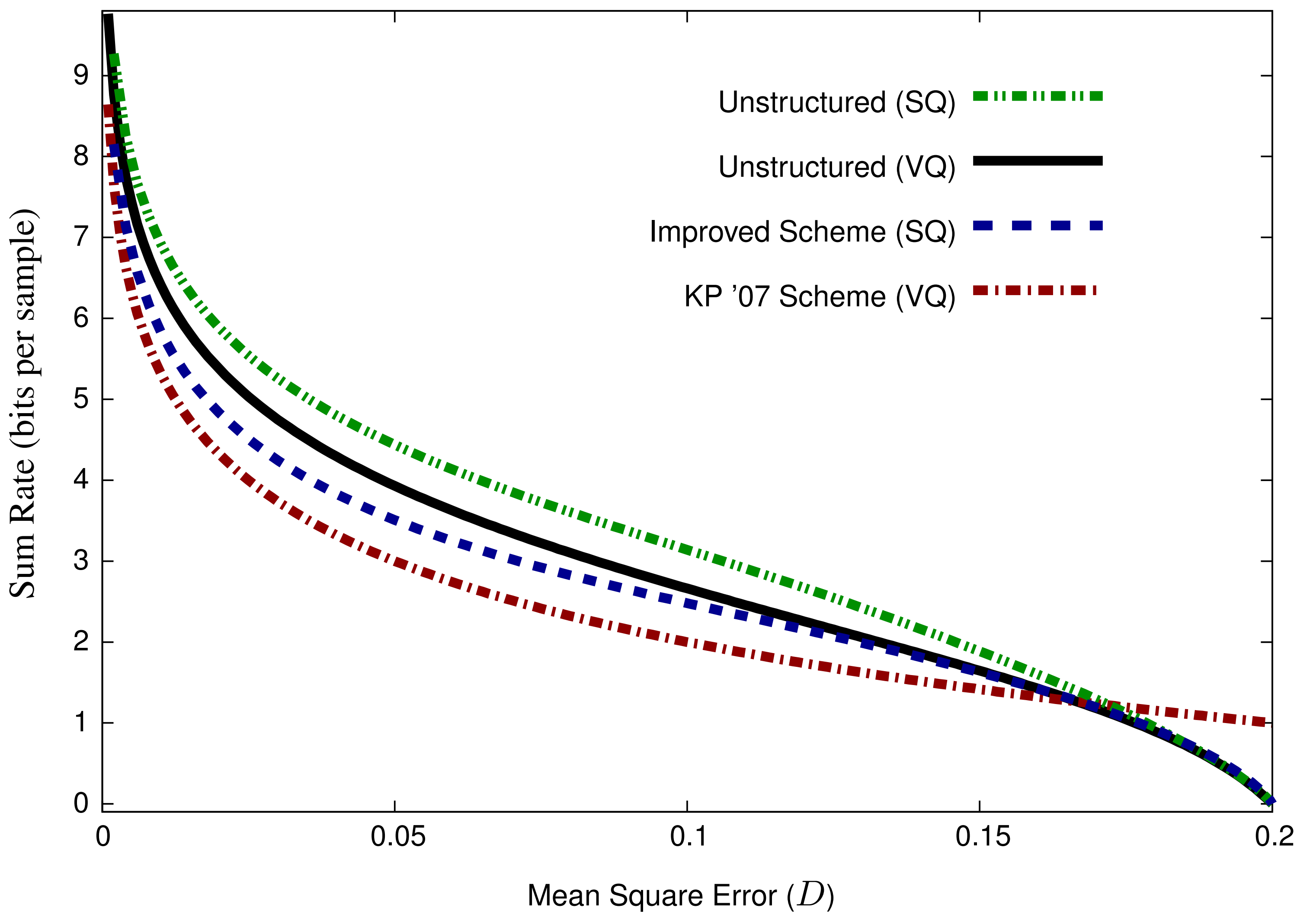
$$R_1 = R_2 = \frac{1}{2} \log \left(\frac{2\text{var}(y_1 - y_2)}{D} \right)$$

The "Warm Grape Juice Problem" [Wagner '11]



- Issue: $\tilde{y}_1^n - \tilde{y}_2^n \mod \Lambda_m$ not uniformly distributed over Λ_m cell
- Fix:
  - View $\tilde{y}_1^n \mod \Lambda_m$ and $\tilde{y}_2^n \mod \Lambda_m$ as supersymbols
  - Apply lossless Körner-Marton trick to them
  - Construction A + Körner-Marton in finite vector spaces
- Good: Closes gap
- Bad: Performance not computable in infinite dimensions

Structured vs. Unstructured



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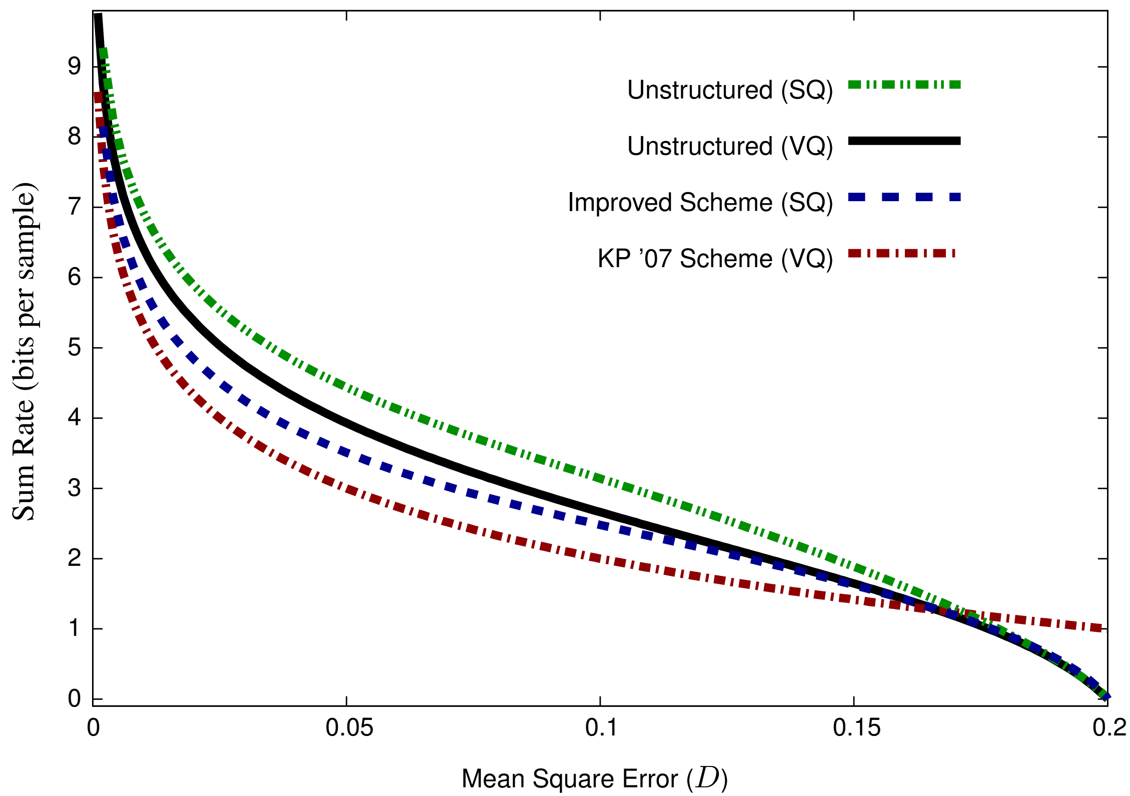
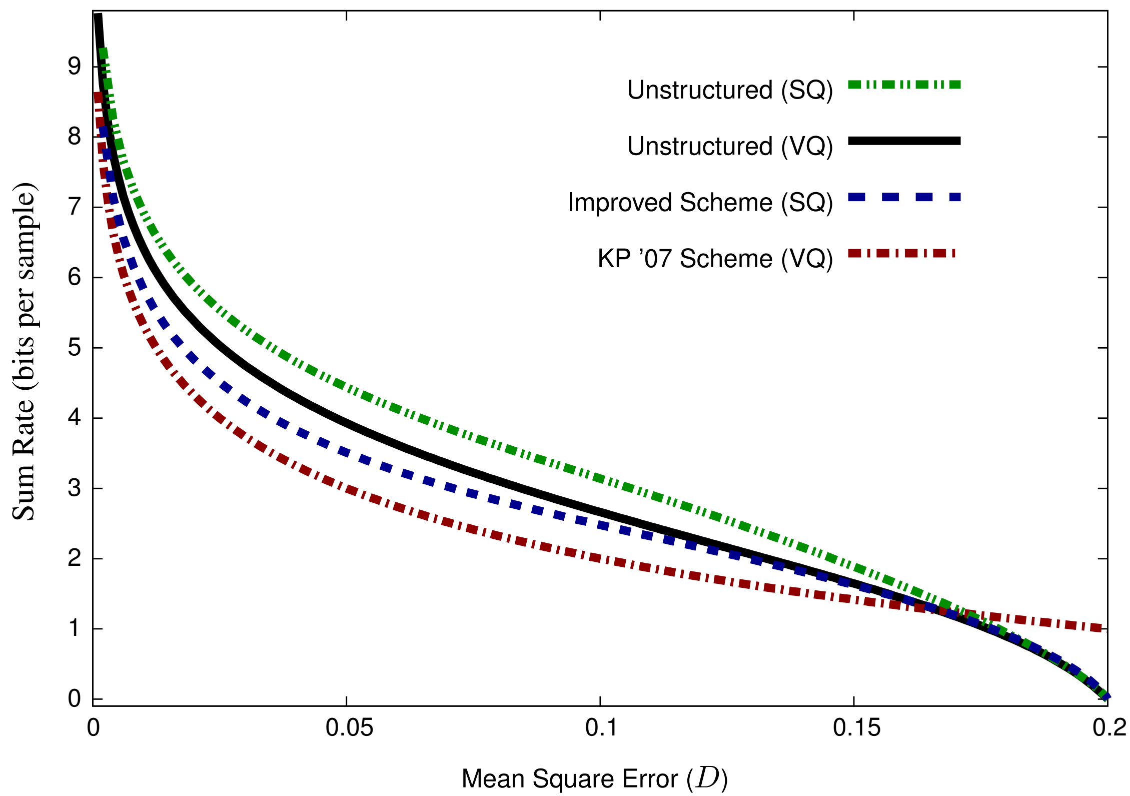


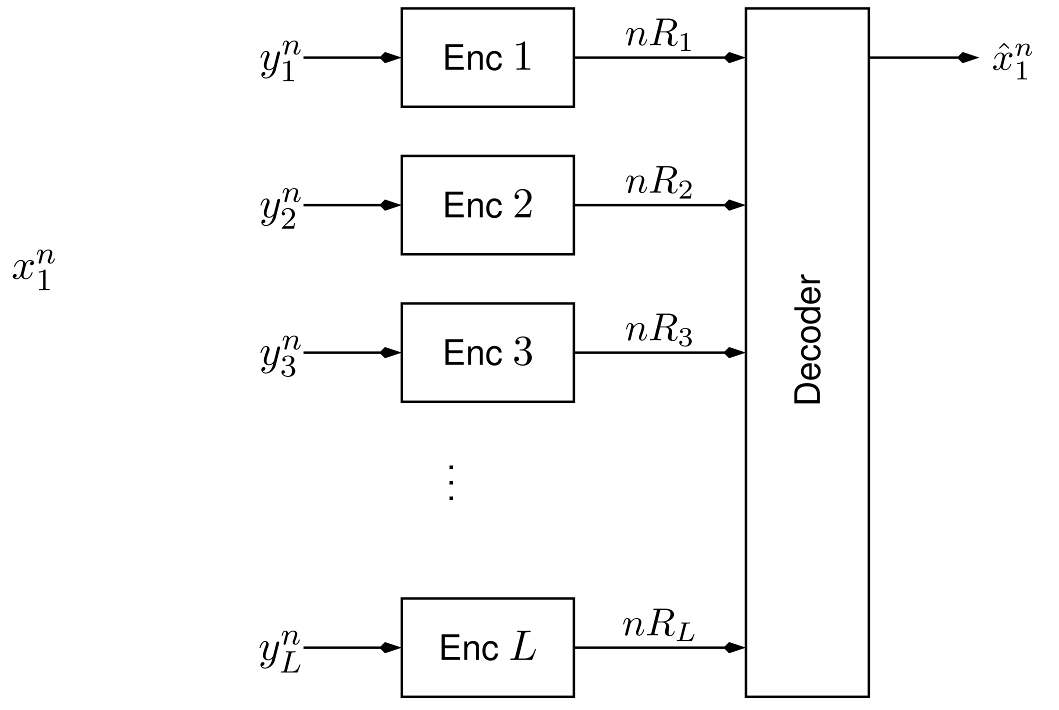
- Open Problem: Can we do better than time-sharing between KP and unstructured?

Structured vs. Unstructured



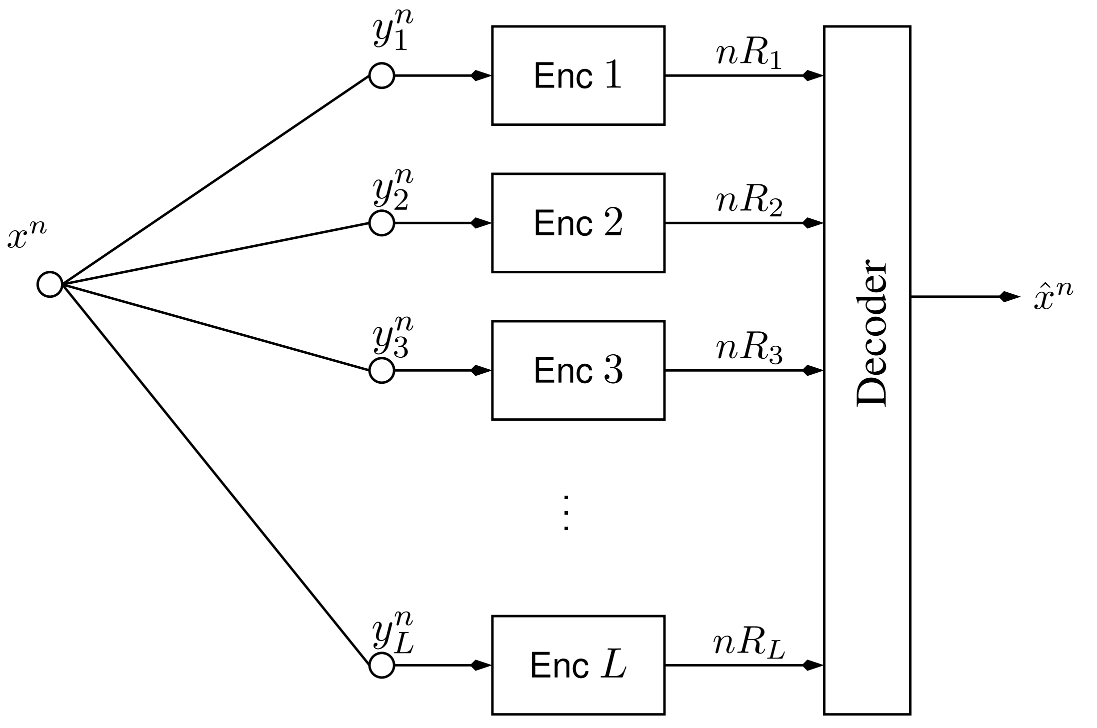
- Extended in different direction by Maddah-Ali and Tse ’10
  - Complete story for two encoders and one reproduction

Many Encoders, One Constraint



$$\frac{1}{n}\sum_{i=1}^{n} E[(x_1(i) - \hat{x}_1(i))^2] \leq d$$

CEO Problem



Theorem (Oohama ’99/’05; Prabhakaran, Tse, and Ramchandran ’04): Unstructured scheme is rate-region optimal

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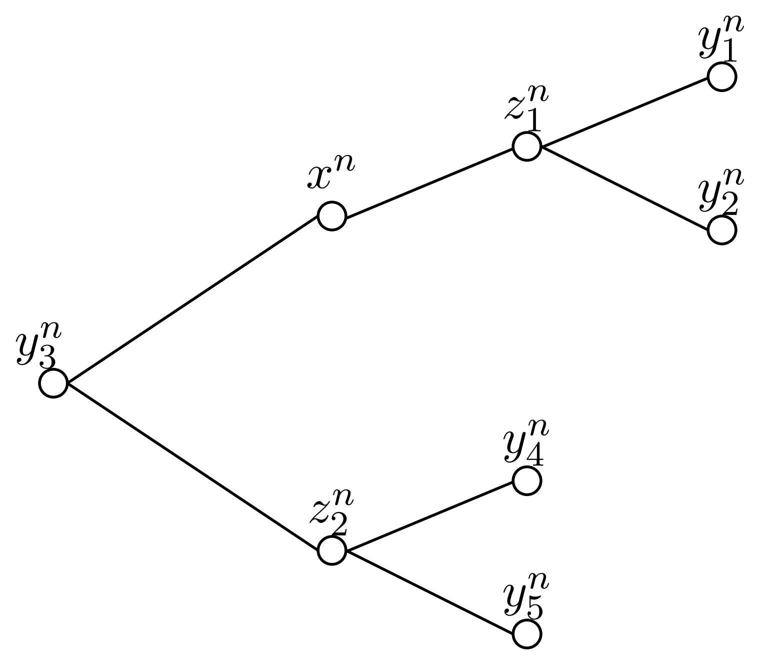

. . . the source variables $y_1, \ldots, y_L, x_1$ can be embedded in a Markov random field that is a tree.

Beyond CEO

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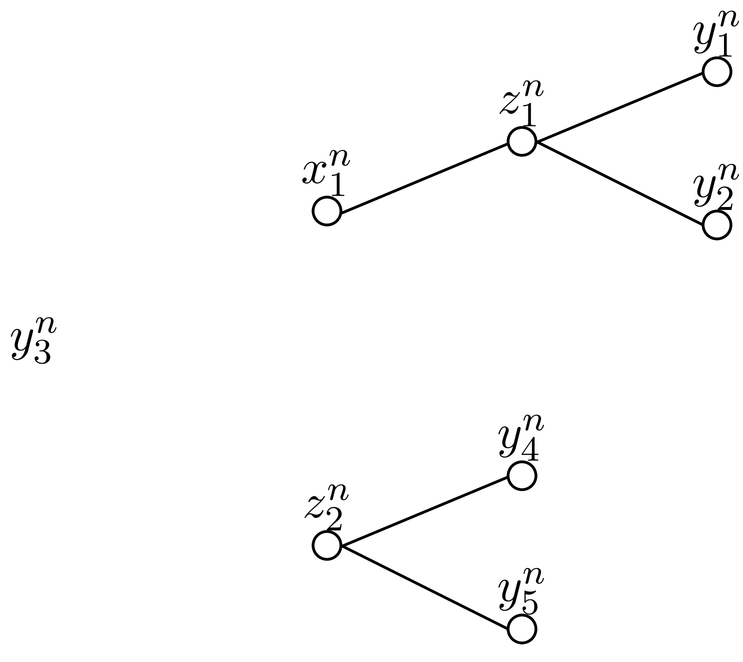

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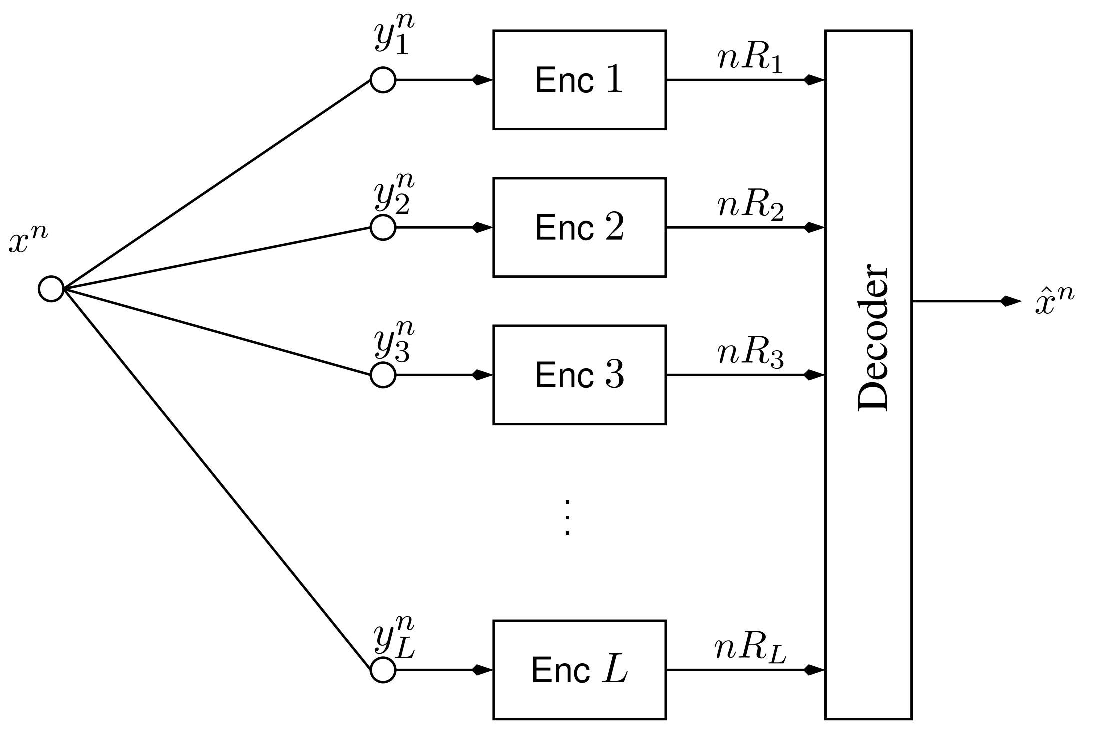
cf. CEO



Improves in two ways:

- Tree can have depth > 1
- May add latent variables to construct the tree

When do Variables Themselves form a Tree?

Theorem: (Speed and Kiiveri ’86): A given set of random variables satisfies the requirements of a given graph iff

implies $\boldsymbol{K}_y^{-1}(i, j) = 0$

Embeddability for Three Variables

- Three variables can be embedded in a Markov tree if and only if

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$$|\rho_{ij}| \geq |\rho_{ik}\rho_{kj}| \quad \text{for all } i, j, \text{ and } k$$

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$$d_{ij} = -\log|\rho_{ij}|$$

Embeddability for Three Variables

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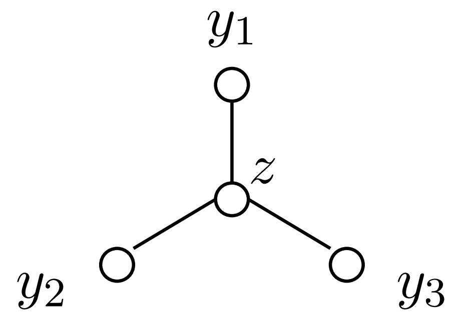
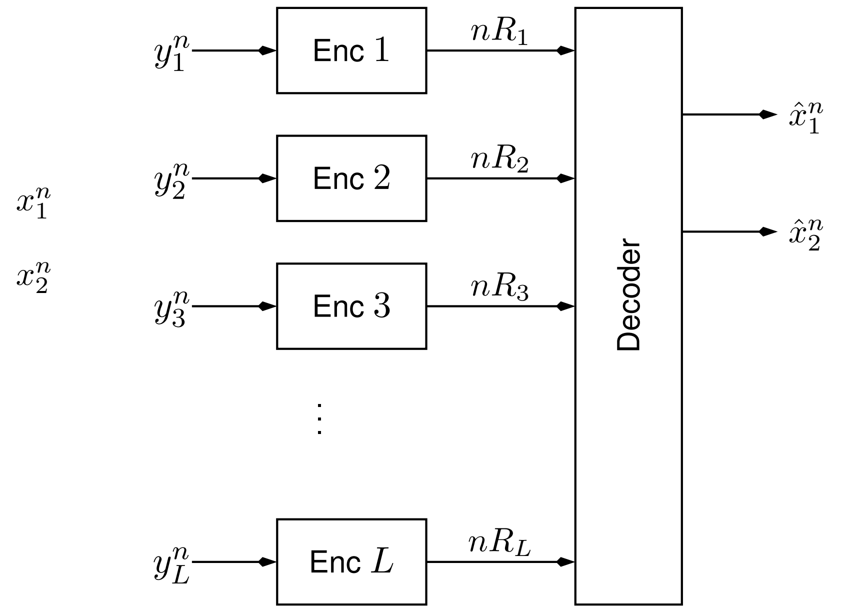


Embeddability for More than Three Variables

- S. Sullivant, “Algebraic geometry of Gaussian Bayesian Networks,” *Adv. Appl. Math.*, 2008
- P. Spirtes *et al.*, *Causation, Prediction, and Search*, MIT Press, 2000

Open Problem: Given $y_1, \ldots, y_L, x_1$, when does there exist $x_2 \equiv x_1$ such that $y_1, \ldots, y_L, x_2$ can be embedded in a Markov tree?

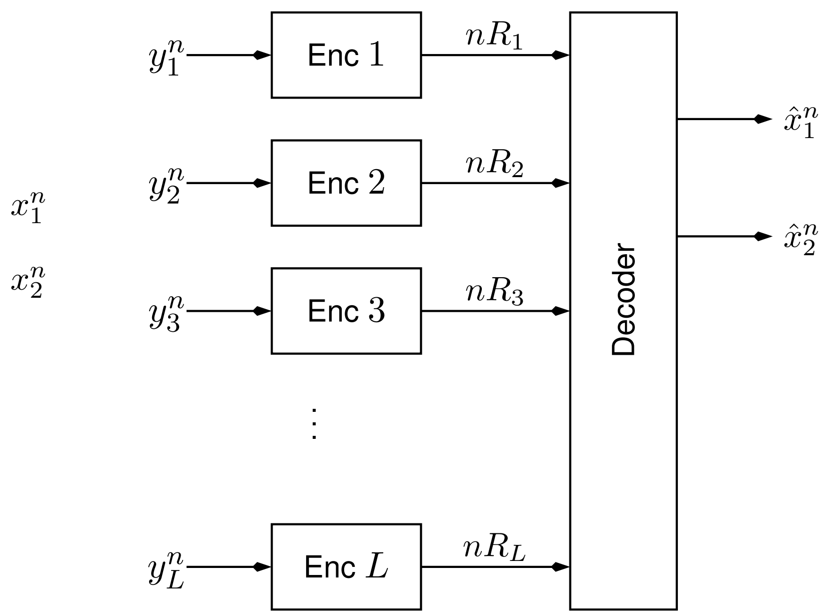
Two Constraints

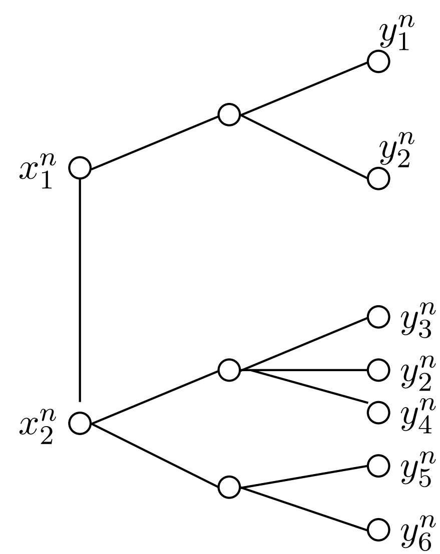


$$\frac{1}{n}\sum_{i=1}^{n} E[(x_1(i) - \hat{x}_1(i))^2] \leq d_1$$

$$\frac{1}{n}\sum_{i=1}^{n} E[(x_2(i) - \hat{x}_2(i))^2] \leq d_2$$

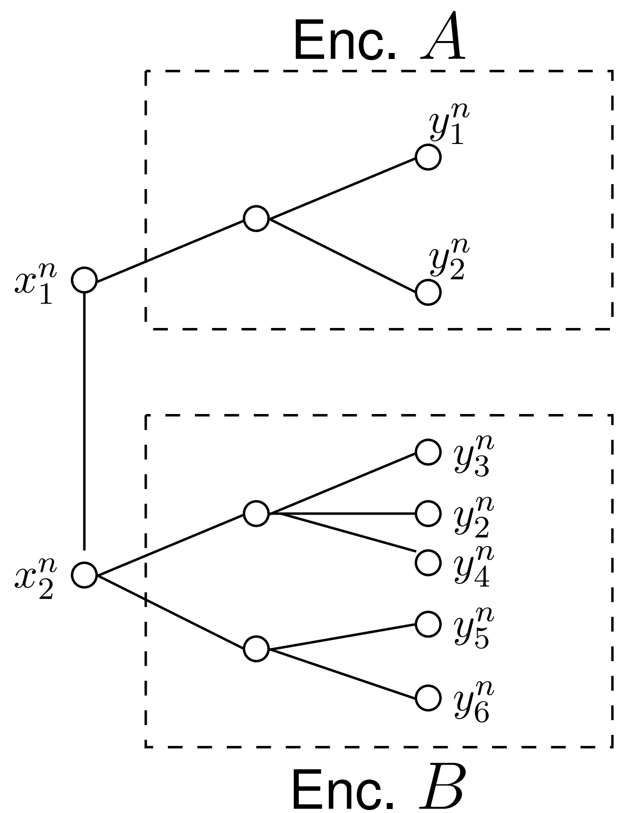
Compressing Neighbors





Proposition (Laourine and Wagner '11): Unstructured scheme is sum rate optimal if the source can be embedded in a depth-two Markov tree with x_1 and x_2 as neighbors.

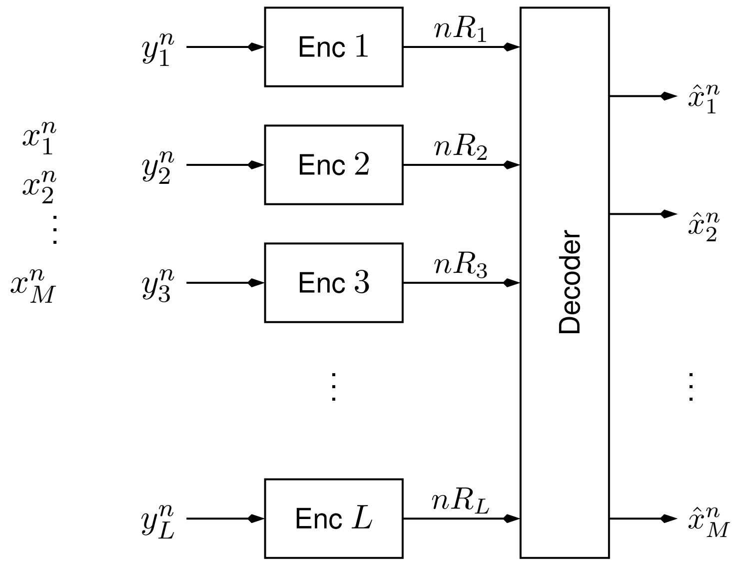
Why Neighbors?



(Sum Rate) $=$ (Enc. A Rate) $+$ (Enc. B Rate)

$+$ (Enc. A Penalty) $+$ (Enc. B Penalty)

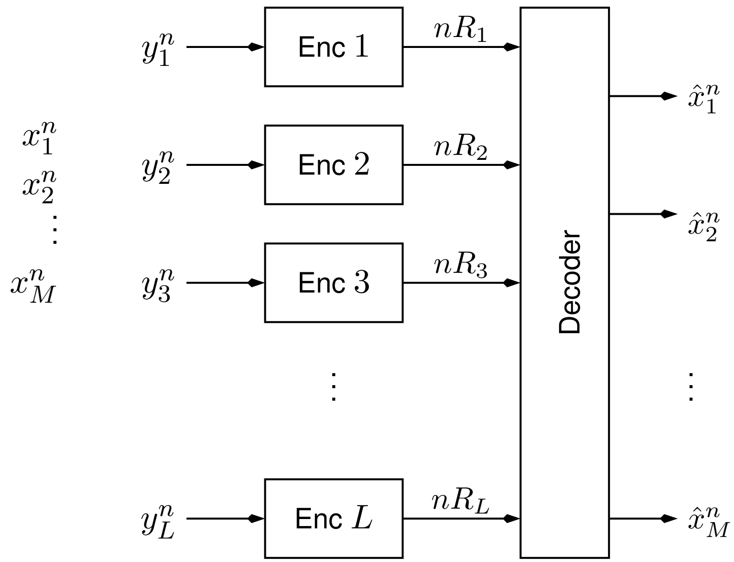
Conclusion



For quadratic Gaussian distributed rate-distortion

- Unstructured scheme is sometimes optimal

Conclusion



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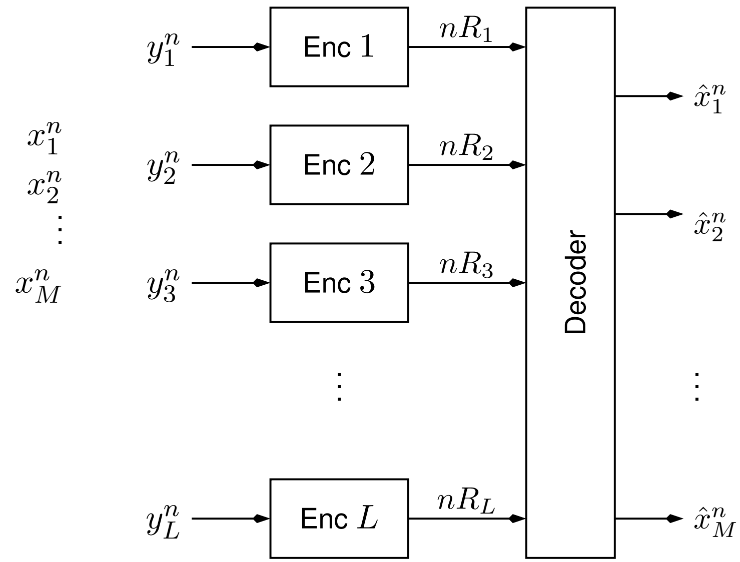

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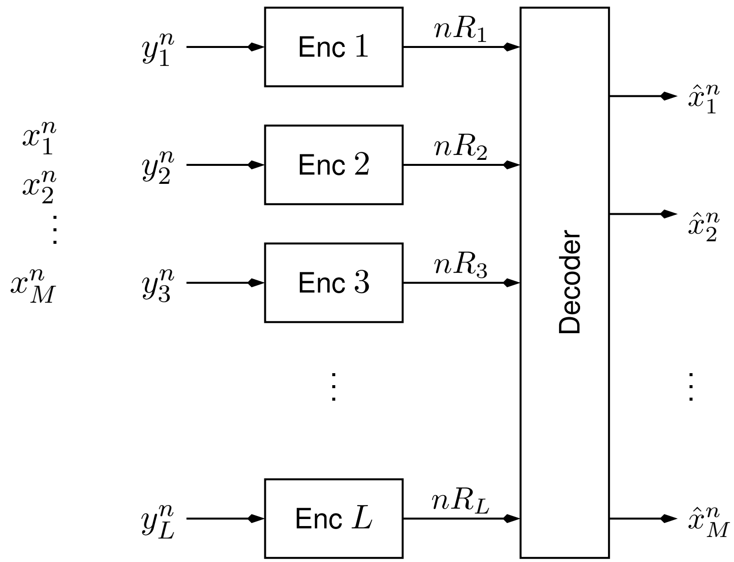

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Embedding Example

$$\boldsymbol{K}_y = \begin{bmatrix} 1 & 1/4 & 1/4 \\ 1/4 & 1 & 1/4 \\ 1/4 & 1/4 & 1 \end{bmatrix}$$

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$$\boldsymbol{K}_y^{-1} = \frac{1}{9} \begin{bmatrix} 10 & -2 & -2 \\ -2 & 10 & -2 \\ -2 & -2 & 10 \end{bmatrix}$$

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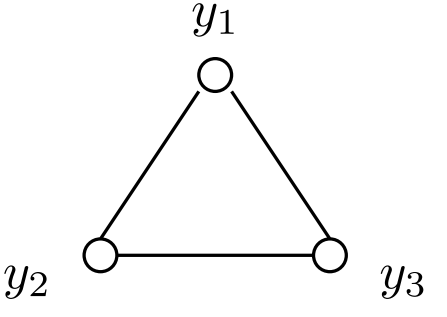

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$$y_1 = \frac{1}{2}y_0 + z_1$$
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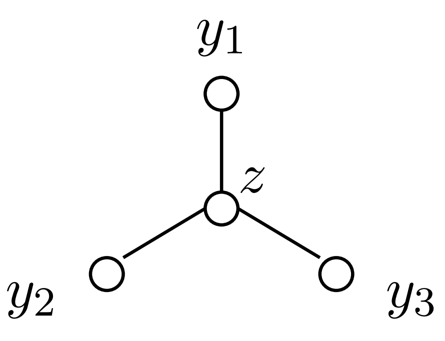

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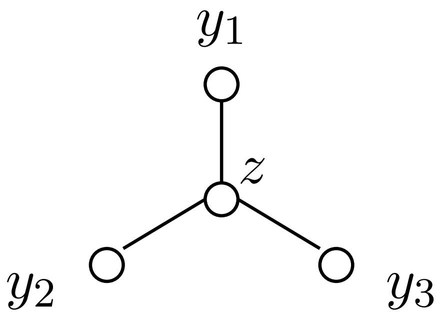

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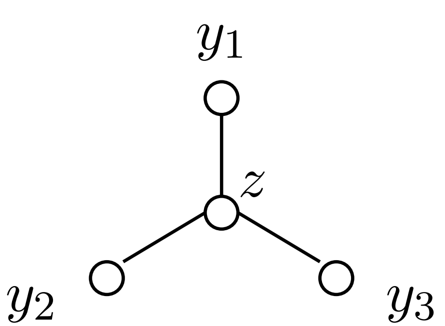

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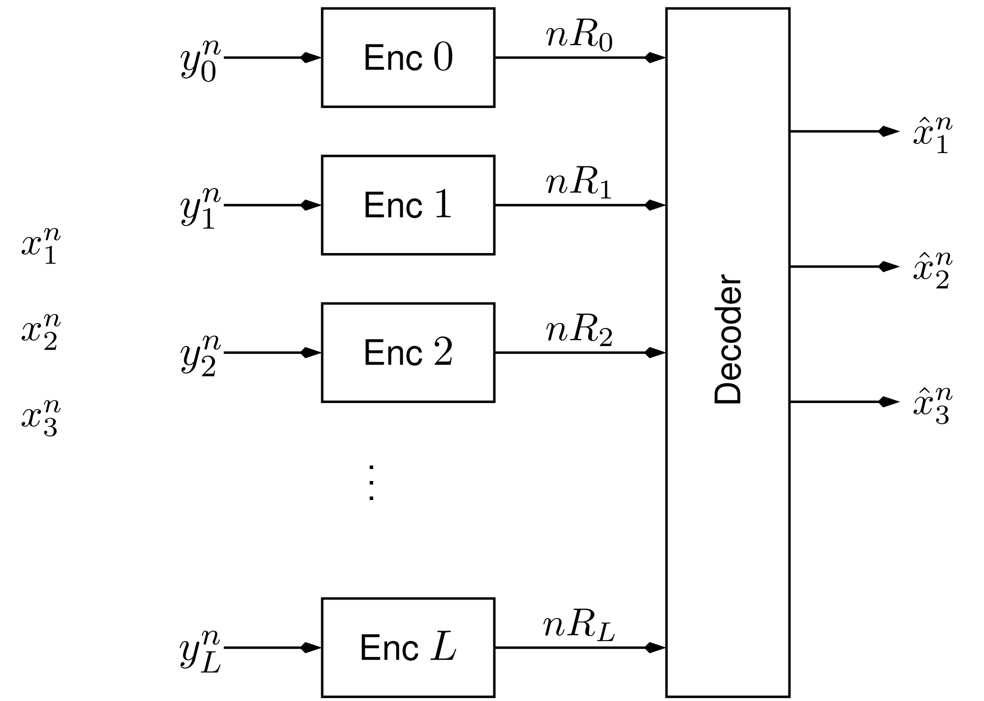
$$\boldsymbol{K}_y^{-1} = \frac{2}{3}\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}$$

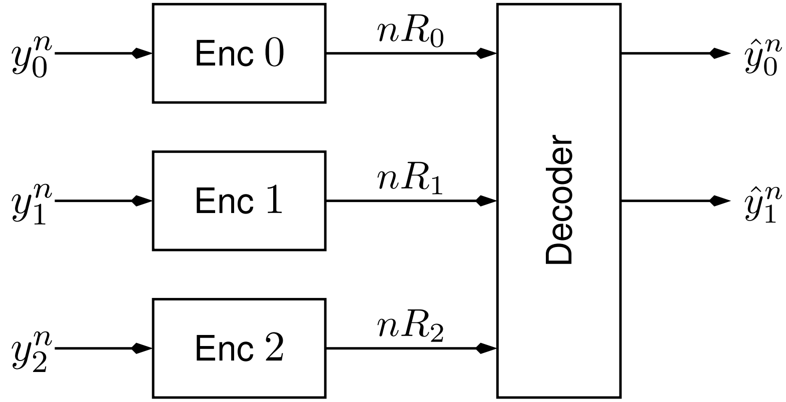
Three Neighbors



Conjecture: Unstructured scheme achieves the entire rate region if the source can be embedded a Markov tree with x_1, x_2, and x_3 as neighbors.

One-Help-Two Problem

Supose y_0, y_1, and y_2 can be embedded in a tree.

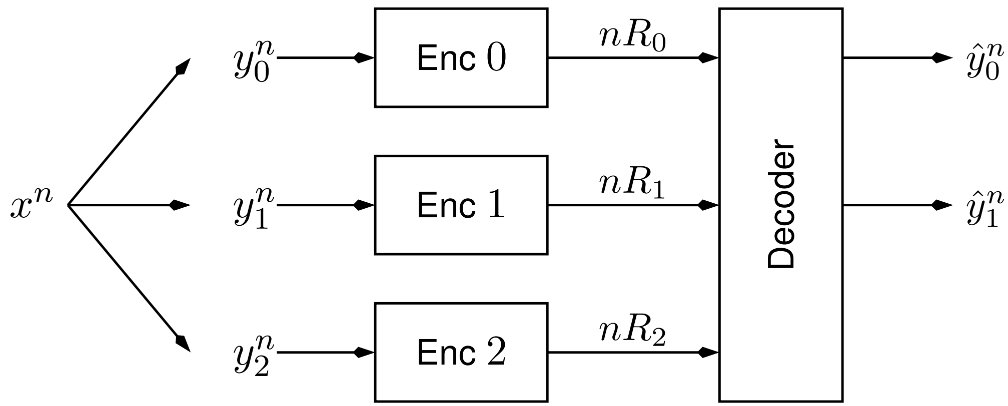


$$\frac{1}{n}\sum_{m=1}^{n} E[(y_0(m) - \hat{y}_0(m))^2] \leq d_0$$

$$\frac{1}{n}\sum_{m=1}^{n} E[(y_1(m) - \hat{y}_1(m))^2] \leq d_1$$

One-Help-Two Problem

Supose y_0, y_1, and y_2 can be embedded in a tree.



$$\frac{1}{n}\sum_{m=1}^{n} E[(y_0(m) - \hat{y}_0(m))^2] \leq d_0$$

$$\frac{1}{n}\sum_{m=1}^{n} E[(y_1(m) - \hat{y}_1(m))^2] \leq d_1$$

$$\frac{1}{n}\sum_{m=1}^{n} E[(x(m) - \hat{x}(m))^2] \leq d$$