Cyclic Orbit Codes

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joint work with F. Manganiello, M. Braun, J. Rosenthal

Notation:

- $\mathcal{G}_q(k,n)$ the Grassmannian
- $\mathcal{U} \in \mathcal{G}_q(k, n)$ a vector space
- $U \in Mat_{k \times n}$ its matrix representation
- GL_n the general linear group
- $A \in GL_n$ a matrix
- $G \leq GL_n$ a subgroup
- group operation from the right on $\mathcal{G}_q(k, n)$:

$$\begin{array}{cccc} \mathcal{G}_q(k,n) \times GL_n & \longrightarrow & \mathcal{G}_q(k,n) \\ (\mathcal{U},A) & \longmapsto & \mathcal{U}A := \operatorname{row} \operatorname{space}(UA) \end{array}$$

Definition

Let $\mathcal{U} \in \mathcal{G}_q(k, n)$ be fixed and G a subgroup of GL_n . Then

 $\mathcal{U}G:=\{\mathcal{U}A\mid A\in G\}$

is called an *orbit code* .

If an orbit code can be defined by a cyclic subgroup $G \leq GL_n$, it is called a *cyclic* orbit code.

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Theorem

Let $C = \{UA \mid A \in G\}$ be an orbit code. Then

$$d_{\min}(\mathcal{C}) = \min\{d_S(\mathcal{U}A, \mathcal{U}A') \mid A, A' \in G/Stab(\mathcal{U})\}\$$

= min{ $d_S(\mathcal{U}, \mathcal{U}A) \mid A \in G/Stab(\mathcal{U})\}.$

("Linearity")

Theorem

Let $G \leq GL_n$ and $H = S^{-1}GS$ for some $S \in GL_n$. Moreover, let $\mathcal{U} \in \mathcal{G}_q(k, n)$ and $\mathcal{V} := \mathcal{U}S$. Then the conjugate orbit codes

$$\mathcal{C} := \{ \mathcal{U}A | A \in G \} \text{ and } \mathcal{C}' := \{ \mathcal{V}B | B \in H \}$$

have the same cardinality and minimum distance. ("Equivalence")

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Cyclic case: Any matrix is conjugate to its rational canonical form.

 \implies It is sufficient to investigate the orbits of groups generated by rational canonical forms!!!

The results are then carried over to any irreducible cyclic orbit code via the choice of starting point of the orbit.

Spread codes are constant dimension codes with minimum distance 2k ("no intersection") and cardinality $\frac{q^n-1}{q^k-1}$ (covering the whole space).

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If $k|n, c := \frac{q^n - 1}{q^k - 1}$ and α a primitive element of \mathbb{F}_{q^n} , then the vector space generated by $1, \alpha^c, ..., \alpha^{(k-1)c}$ is equal to $\{\alpha^{ic}|i=0, ..., q^k - 2\} \cup \{0\} = \mathbb{F}_{q^k}.$

Theorem

The set

$$\mathcal{S} = \left\{ \alpha^i \cdot \mathbb{F}_{q^k} \mid i = 0, \dots, c - 1 \right\}$$

defines a spread code in $\mathbb{F}_{q^n} \cong \mathbb{F}_q^n$.

Example (Spread of $\mathcal{G}_2(2,4)$):

•
$$c = \frac{q^n - 1}{q^k - 1} = \frac{15}{3} = 5$$

•
$$p(x) := x^4 + x + 1$$
 primitive

- α a root of p(x)
- *P* its companion matrix

•
$$\phi : \mathbb{F}_q^n \to \mathbb{F}_{q^n}$$
 vector space isomorphism

$$u_1 = \phi^{-1}(\alpha^0) = \phi^{-1}(1) = (1000)$$

$$u_2 = \phi^{-1}(\alpha^c) = \phi^{-1}(\alpha^5) = \phi^{-1}(\alpha^2 + \alpha) = (0110)$$

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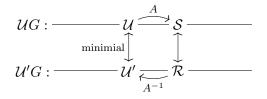
Then the following orbit is a spread code:

$$\mathcal{U}\langle P \rangle = \operatorname{rs} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right] \left\langle \left(\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{array} \right) \right\rangle$$

Minimum distance decoder:

For an orbit code $\mathcal{C} = \mathcal{U}G$ and a received subspace $\mathcal{R} \in \mathcal{P}(\mathbb{F}_q^n)$ the minimum distance decoder searches for the group element $A \in G$ such that $\dim(\mathcal{R} \cap \mathcal{U}A)$ is maximal.

Syndrome decoding:



Question: Are there groups where it is easy to find the "syndrome" and the orbit a given element is on?

Let p(x) be primitive and P its companion matrix. $\implies \langle P \rangle$ acts transitively on \mathbb{F}_q^n

INPUT: Code $C = \mathcal{U}\langle P \rangle$, received \mathcal{R} OUTPUT: $A \in \langle P \rangle$ such that dim $(\mathcal{R} \cap \mathcal{U}A)$ is maximal set $d := 0, A := I_{n \times n}$ for all $v \in \mathcal{U} \setminus \{0\}$ do for all $w \in \mathcal{R} \setminus \{0\}$ do compute $A' := \phi^{-1}(\phi(w)\phi(v)^{-1})$ compute $d' := \dim(\mathcal{R} \cap \mathcal{U}A)$ if d' > d then set d := d' and A := A'return A Let p(x) be primitive and P its companion matrix. $\implies \langle P \rangle$ acts transitively on \mathbb{F}_q^n

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Complexity: $\mathcal{O}(q^{k+\dim(\mathcal{R})}n(k+\dim(\mathcal{R}))^2)$ over \mathbb{F}_q .

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Thank you!