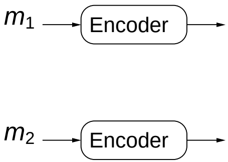
The Degrees of Freedom of Compute-and-Forward

Urs Niesen
Jointly with Phil Whiting

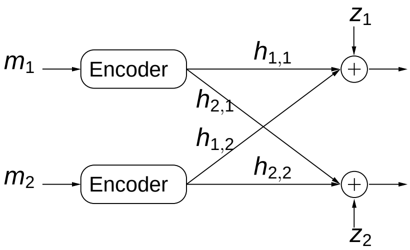
Bell Labs, Alcatel-Lucent

Problem Setting



- K transmitters, messages $m_1, \ldots, m_K$, power constraint P

Problem Setting



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- Gaussian channel with constant gains $\boldsymbol{H} = \{h_{\ell,k}\}$

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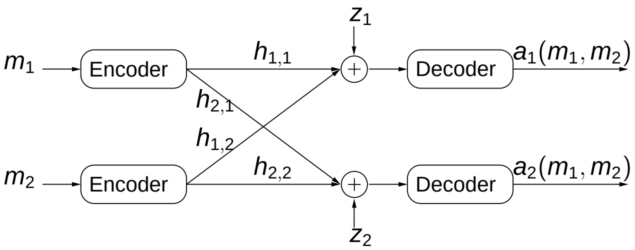

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- Decode any deterministic function a_ℓ of the messages $m_1, \ldots, m_K$

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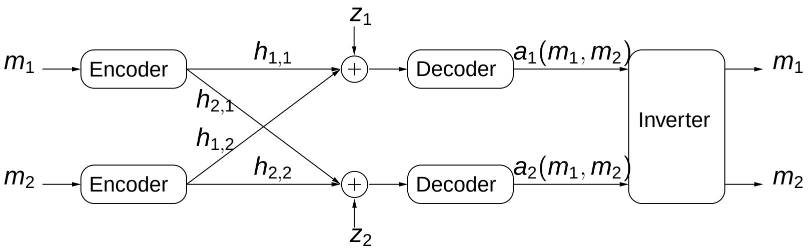

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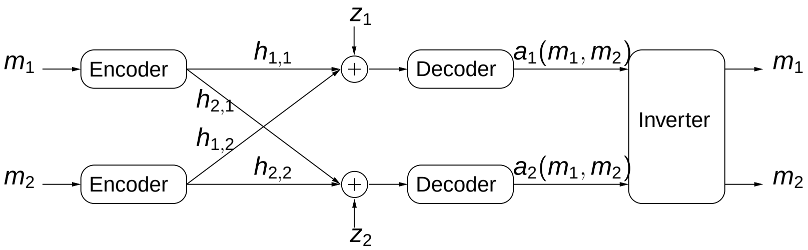
- Gaussian channel with constant gains $\boldsymbol{H} = \{h_{\ell,k}\}$
- Decode any deterministic function a_ℓ of the messages $m_1, \ldots, m_K$
- Invert all computed functions $a_1, \ldots, a_K$ to recover the messages $m_1, \ldots, m_K$

Problem Setting



- Computation capacity $C(P, \boldsymbol{H}, \{a_\ell\})$ for fixed function $\{a_\ell\}$

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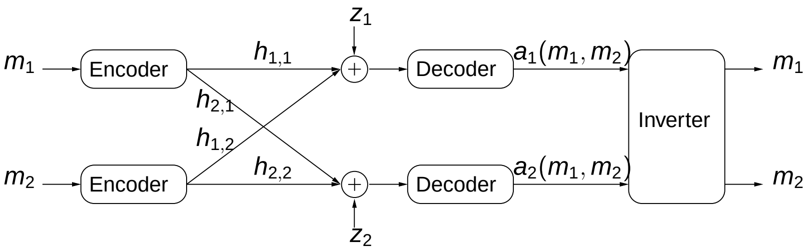

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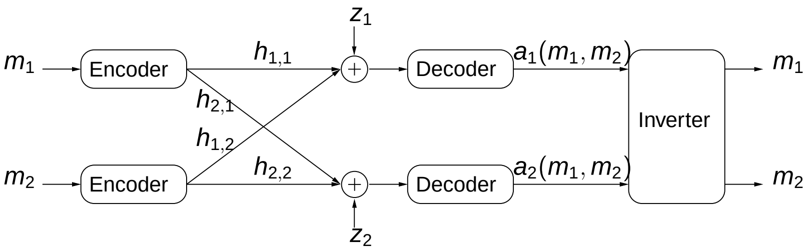

- Computation capacity $C(P, \boldsymbol{H}, \{a_\ell\})$ for fixed function $\{a_\ell\}$
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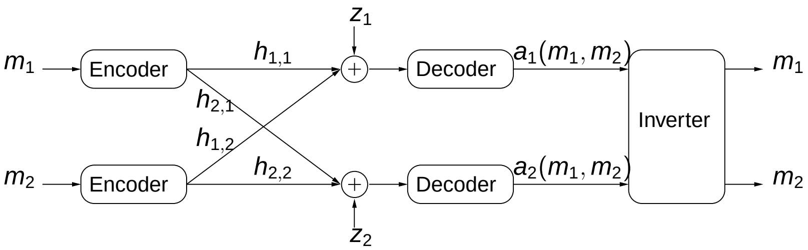
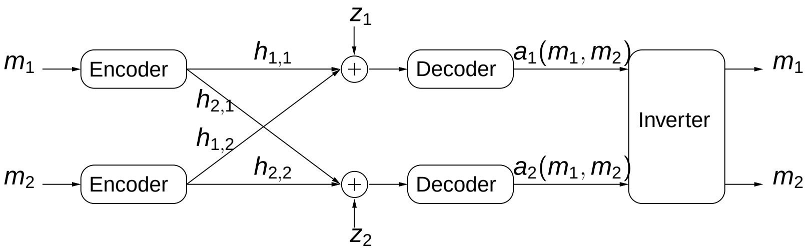
- Computation capacity

$$C(P, \boldsymbol{H}) \triangleq \max_{\{a_\ell\}} C(P, \boldsymbol{H}, \{a_\ell\})$$

with maximization over all invertible functions $\{a_\ell\}$



- Identity function $\boldsymbol{A} = \boldsymbol{I} \Rightarrow C(P, \boldsymbol{H}, \boldsymbol{I})$ is the capacity of the K-user interference channel



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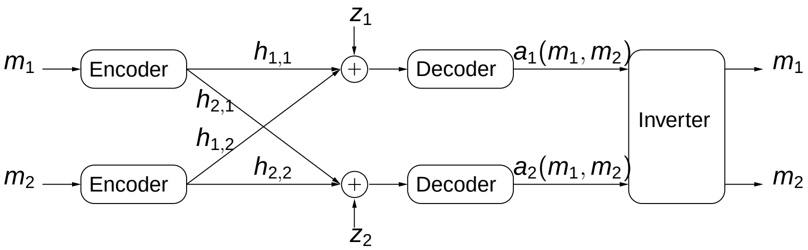
$$C(P, \boldsymbol{H}) \geq C(P, \boldsymbol{H}, \boldsymbol{I})$$

Computation Capacity

Some Special Cases and Bounds



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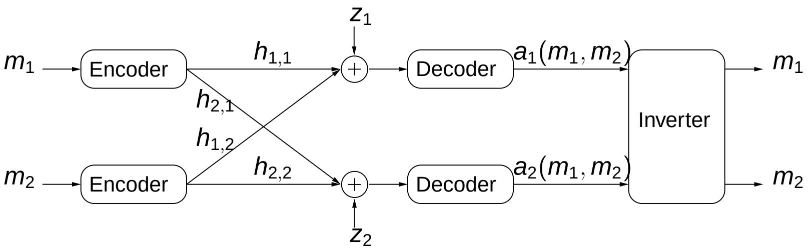
$$C(P, \boldsymbol{H}) \geq C(P, \boldsymbol{H}, \boldsymbol{I}) = \frac{K}{4} \log(P) + o(\log(P))$$



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Computation Capacity
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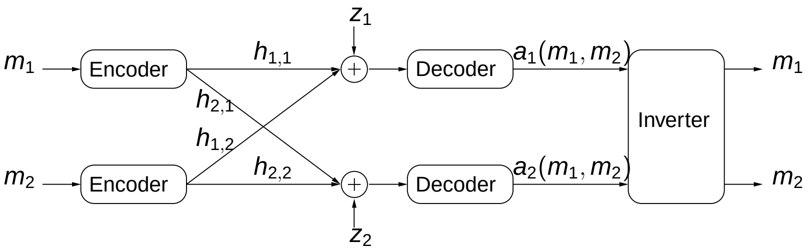

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- Allow cooperation among transmitters and among receivers $\Rightarrow K \times K$ MIMO channel

Computation Capacity

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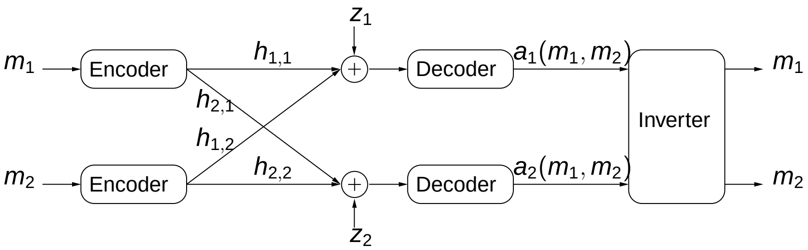

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Computation Capacity

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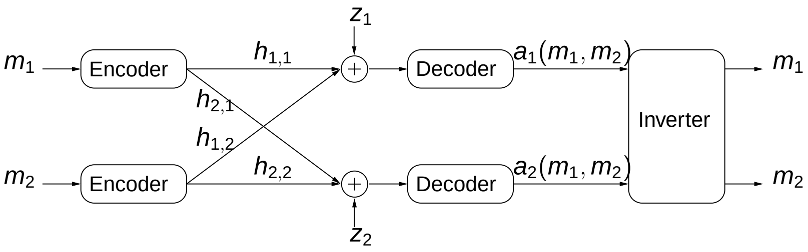

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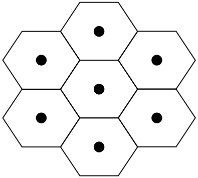
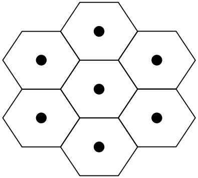
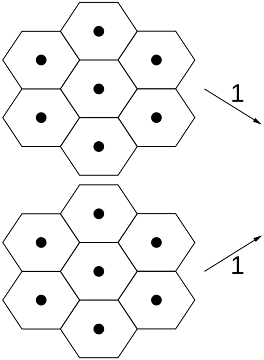




- Two transmitters, one receiver, $\boldsymbol{h} = (1, 1)$



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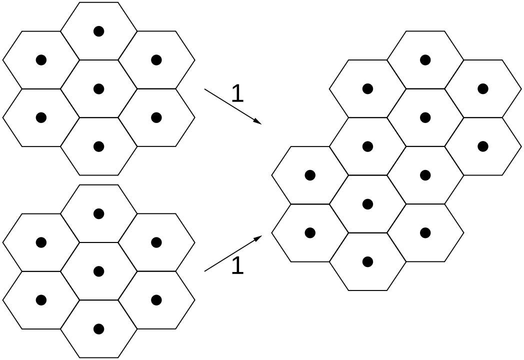

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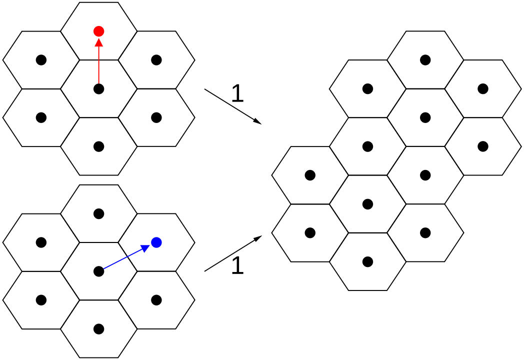
Computation Capacity

Lattice Codes

Nazer and Gastpar (2009)



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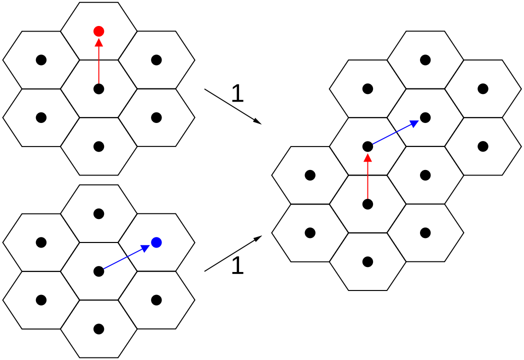

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Computation Capacity
Lattice Codes

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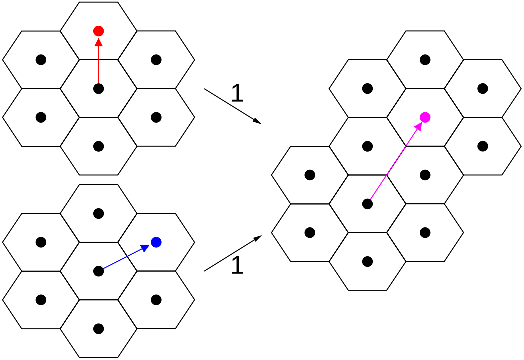
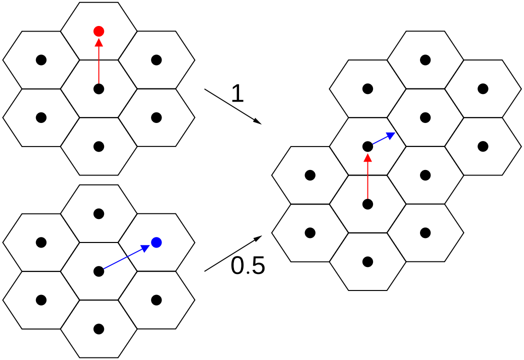
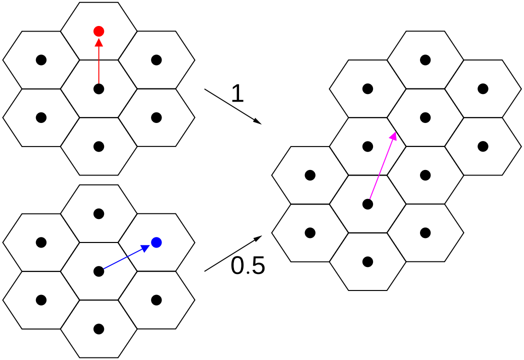


- Two transmitters, one receiver, $\boldsymbol{h} = (1, 1)$
- Decode $\boldsymbol{a} = (1, 1)$



- Two transmitters, one receiver, $\boldsymbol{h} = (1, 0.5)$



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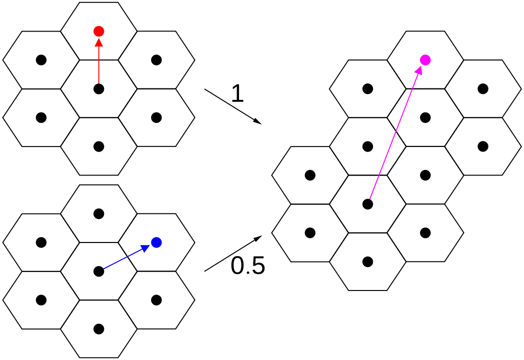
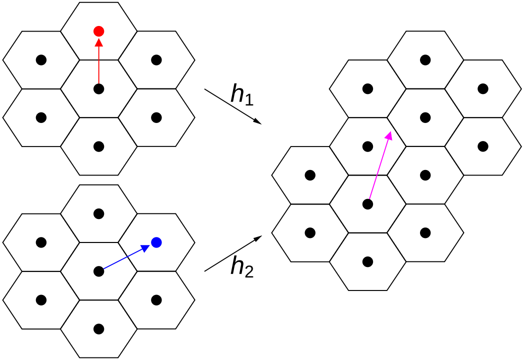
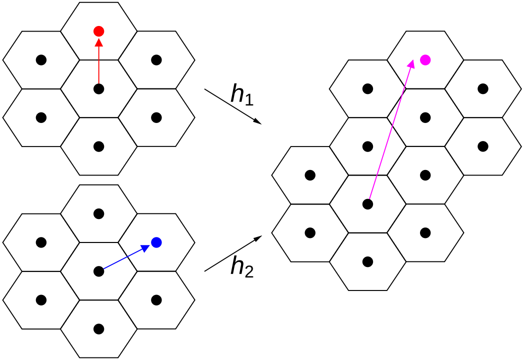


- Two transmitters, one receiver, $\boldsymbol{h} = (1, 0.5)$
- Scale output by $\beta = 2$, decode $\boldsymbol{a} = (2, 1)$



- K transmitters, one receiver, $\boldsymbol{h} \in \mathbb{R}^K$



- K transmitters, one receiver, $\boldsymbol{h} \in \mathbb{R}^K$
- Scale output by $\beta \in \mathbb{R}$, decode $\boldsymbol{a} \in \mathbb{Z}^K$

Computation Capacity

Lattice Codes

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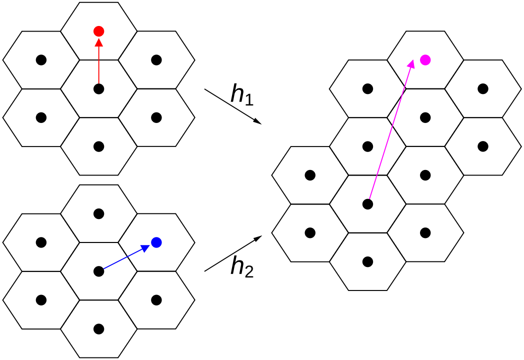

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- Scale output by $\beta \in \mathbb{R}$, decode $\boldsymbol{a} \in \mathbb{Z}^K$

$$C(P, \boldsymbol{h}, \boldsymbol{a}) \geq \max_{\beta} \frac{1}{2} \log \left(\frac{P}{\beta^2 + P \|\beta \boldsymbol{h} - \boldsymbol{a}\|^2} \right)$$

Computation Capacity
Lattice Codes

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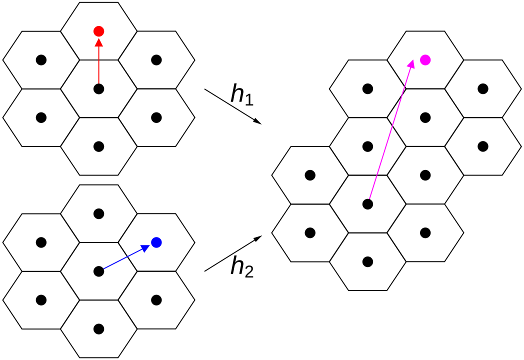

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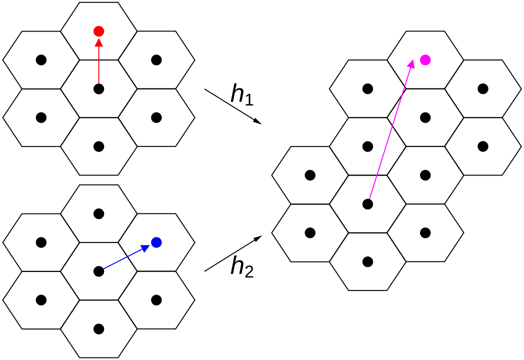

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Computation Capacity

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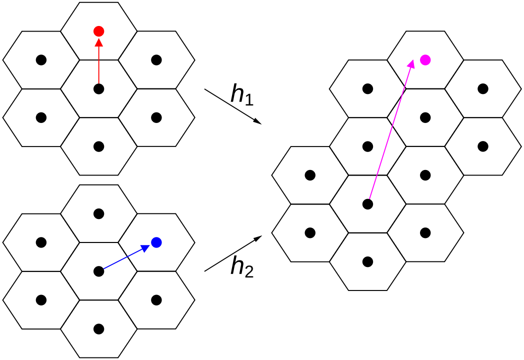
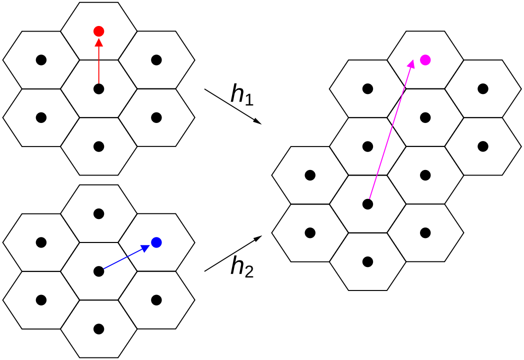


- K transmitters, K receivers, $\boldsymbol{H} \in \mathbb{R}^{K \times K}$
- Decode $\boldsymbol{A} \in \mathbb{Z}^{K \times K}$



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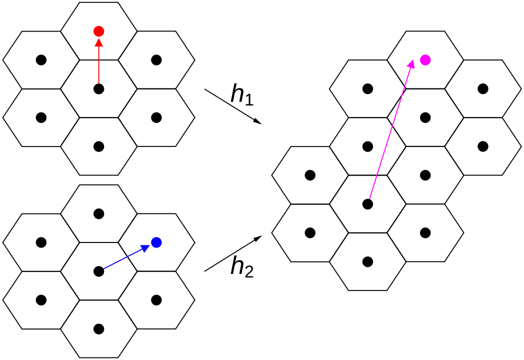

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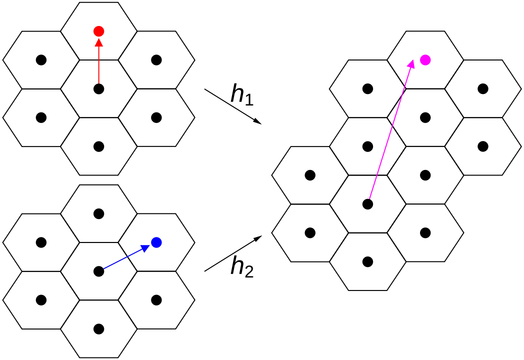
$$\max_{\boldsymbol{A}} C(P, \boldsymbol{H}, \boldsymbol{A}) \geq \max_{\boldsymbol{A}} R_L(P, \boldsymbol{H}, \boldsymbol{A})$$



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$$C(P, \boldsymbol{H}) \geq R_L(P, \boldsymbol{H})$$

Questions

- We already know that the computation capacity satisfies

$$K/2 \leq \lim_{P\to\infty} \frac{C(P, \boldsymbol{H})}{\frac{1}{2}\log(P)} \leq K$$

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- What are the degrees of freedom of compute-and-forward

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$$\lim_{P\to\infty} \frac{C(P,\boldsymbol{H})}{\frac{1}{2}\log(P)} = ?$$

- What are the degrees of freedom achieved by lattice codes

$$\lim_{P\to\infty} \frac{R_L(P,\boldsymbol{H})}{\frac{1}{2}\log(P)} = ?$$

Performance of Lattice Codes

$$R_L(P, \boldsymbol{h}, \boldsymbol{a}) \triangleq \frac{1}{2} \log\left(\frac{1 + P\|\boldsymbol{h}\|^2}{\|\boldsymbol{a}\|^2 + P\left(\|\boldsymbol{a}\|^2\|\boldsymbol{h}\|^2 - (\boldsymbol{h}^T\boldsymbol{a})^2\right)} \right)$$

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- Integer channel gains $\boldsymbol{H} \in \mathbb{Z}^{K \times K} \Rightarrow$ Set $\boldsymbol{A} = \boldsymbol{H} \in \mathbb{Z}^{K \times K}$

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Performance of Lattice Codes

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- Rational channel gains $\boldsymbol{H} \in \mathbb{Q}^{K \times K} \Rightarrow$ Set $\boldsymbol{A} = q\boldsymbol{H} \in \mathbb{Z}^{K \times K}$

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Performance of Lattice Codes

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Performance of Lattice Codes

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- Real channel gains $\boldsymbol{H} \in \mathbb{R}^{K\times K} \Rightarrow$?

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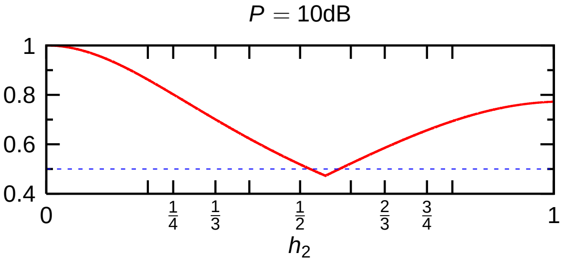
Performance of Lattice Codes



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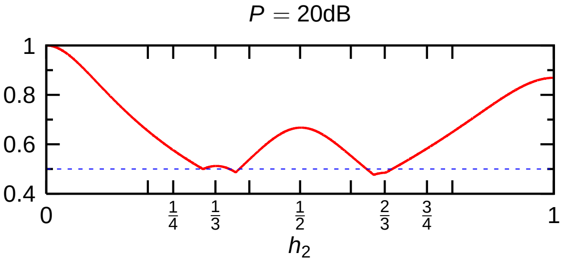
$$\frac{\max_{\boldsymbol{a}} R_L(P, \boldsymbol{h}, \boldsymbol{a})}{\frac{1}{2}\log(1 + \|\boldsymbol{h}\|^2 P)}$$

Performance of Lattice Codes



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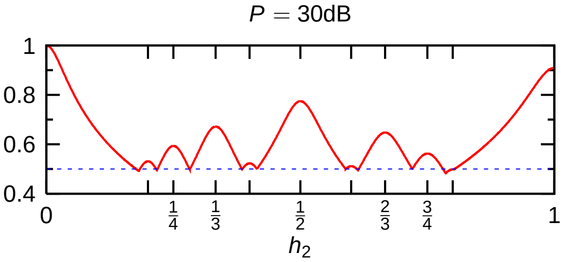

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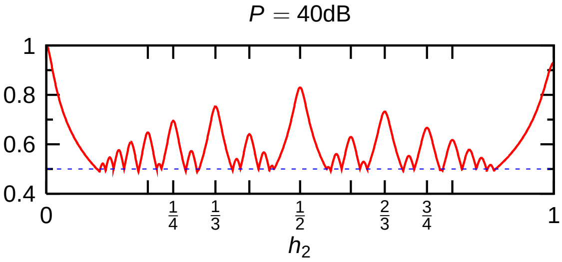

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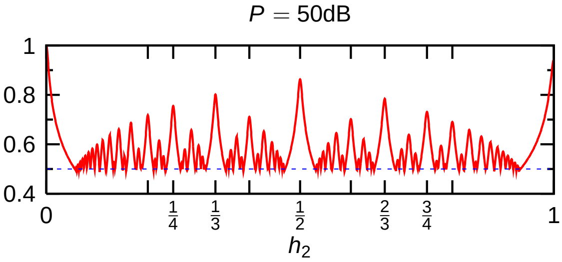

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Compare to:

- MIMO upper bound of K on the degrees of freedom of compute-and-forward
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Is compute-and-forward useful at high SNR?

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- Use a linear outer code to transform noisy linear combinations into noiseless linear combinations

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For any $\varepsilon > 0$, and almost all $(h_1, h_2, \ldots, h_K) \in \mathbb{R}^K$,

$$\min_{q_i \in \mathbb{Z}} \left| h_1 q_1 + \ldots + h_K q_K \right| \geq \Omega\left((\max_i |q_i|)^{1-K-\varepsilon} \right).$$

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- For $A \approx P^{(K-1)/2K}$ and $Q \approx P^{1/2K}$ satisfy power constraint and can remove noise

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- Groshev's Theorem to separate equations
- Linear outer code to drive probability of error to zero

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