Common Invariant Subspaces and Tensor Products for Interference Alignment

(in Wireless Communications and Distributed Storage)

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Collaborations with Prof. Syed A. Jafar, Dr. Cheng Huang, Dr. Jin Li

Algebraic Structure

in Network Information Theory

Algebraic Structure

in Network Information Theory

- Interference Channels (review)
- Distributed Storage (NEW RESULT!)

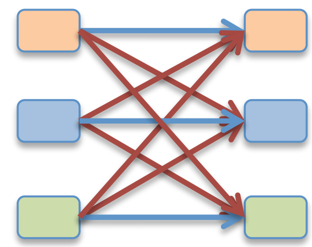
Algebraic Structure

- Common Invariant Subspaces (Structured Codewords)
- Tensor Products , $\otimes$, ("Structured Channels")

in Network Information Theory

- Interference Channels (review)
- Distributed Storage (NEW RESULT!)

Interference Channels



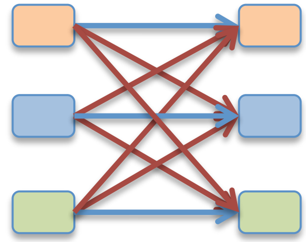
- Random Codes optimal, if interference is “strong” or “noisy”

[Sato 77, 81, Costa-ElGamal 87, Shang-Poor 11, Shang-Kramer-Chen 07, Annapureddy-Veeravalli 08, 09 Shang-Kramer-Chen-Poor 09, Motahari-Khandani 08, Cadambe-Jafar 10]

- Interference Alignment (Structured Coding)
  - Asymptotic Alignment gives K/2 dof in K user channel
  - Lattice Coding yields approximate/exact capacity

[Cadambe-Jafar 07, 08, Bresler-Parekh-Tse 07, Cadambe-Jafar-Shamai 08, Sridharan-Jafarian-Viswhanath-Jafar 08, Sridharan-Jafarian-Vishwanath-Jafar-Shamai 08]

Interference Channels



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Distributed Storage: Exact Repair of MDS Code

[Dimakis et. Al. 08, Wu-Dimakis 09]

(n, k) MDS Code

$\mathbf{A}_1$ $\mathbf{A}_2$ ------ $\mathbf{A}_k$ $\mathbf{f}_{k+1}(\mathbf{A}_1, \ldots, \mathbf{A}_k)$ ----- $\mathbf{f}_n(\mathbf{A}_1, \ldots, \mathbf{A}_k)$

Distributed Storage: Exact Repair of MDS Code

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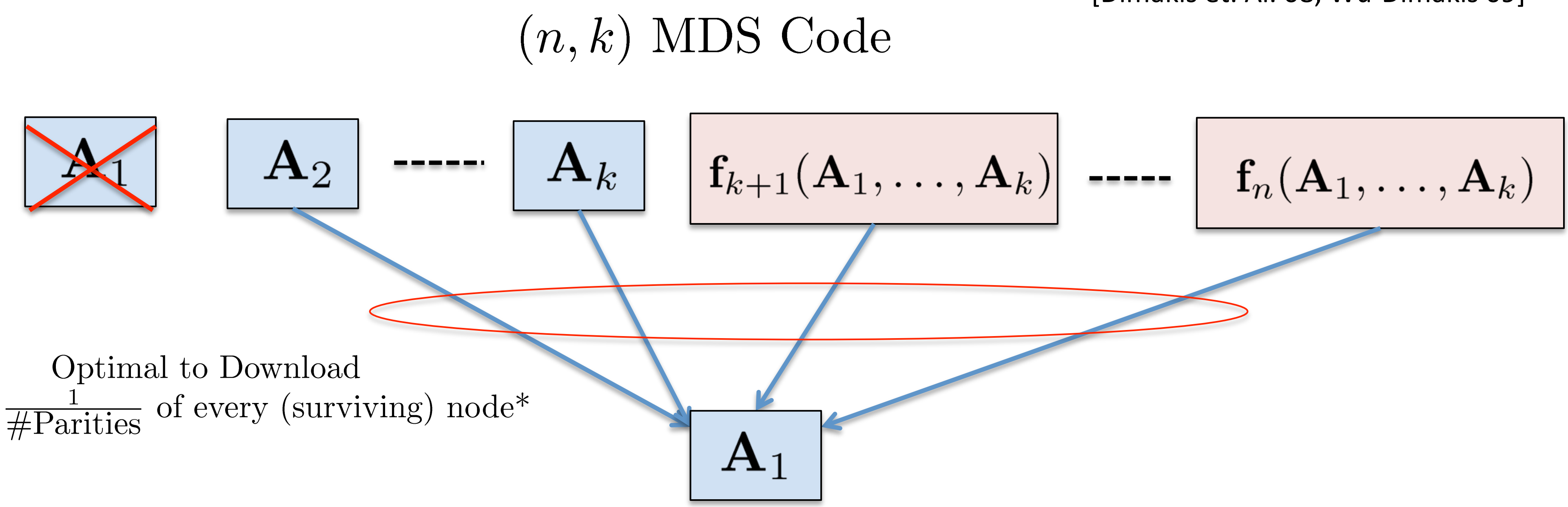
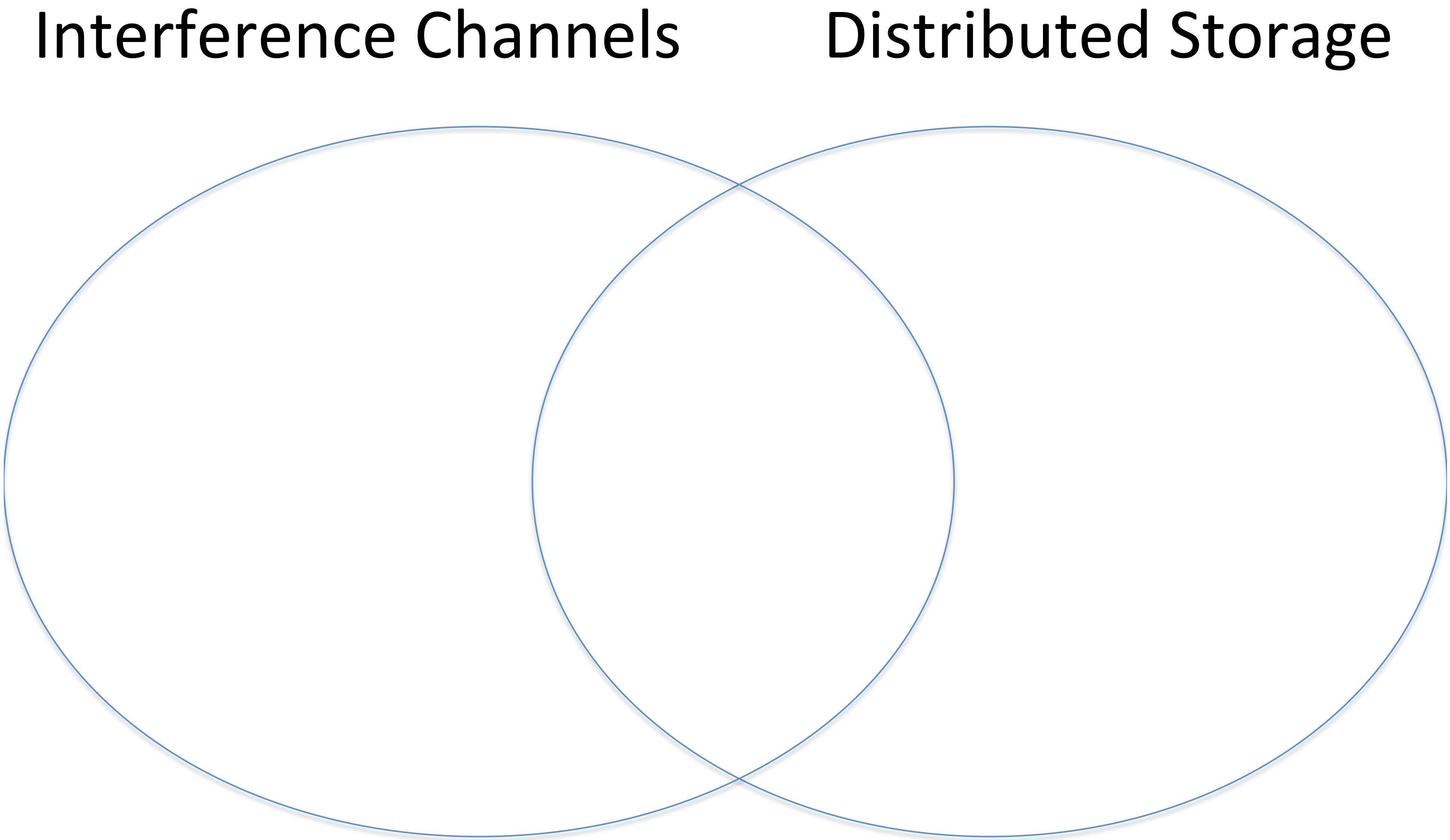
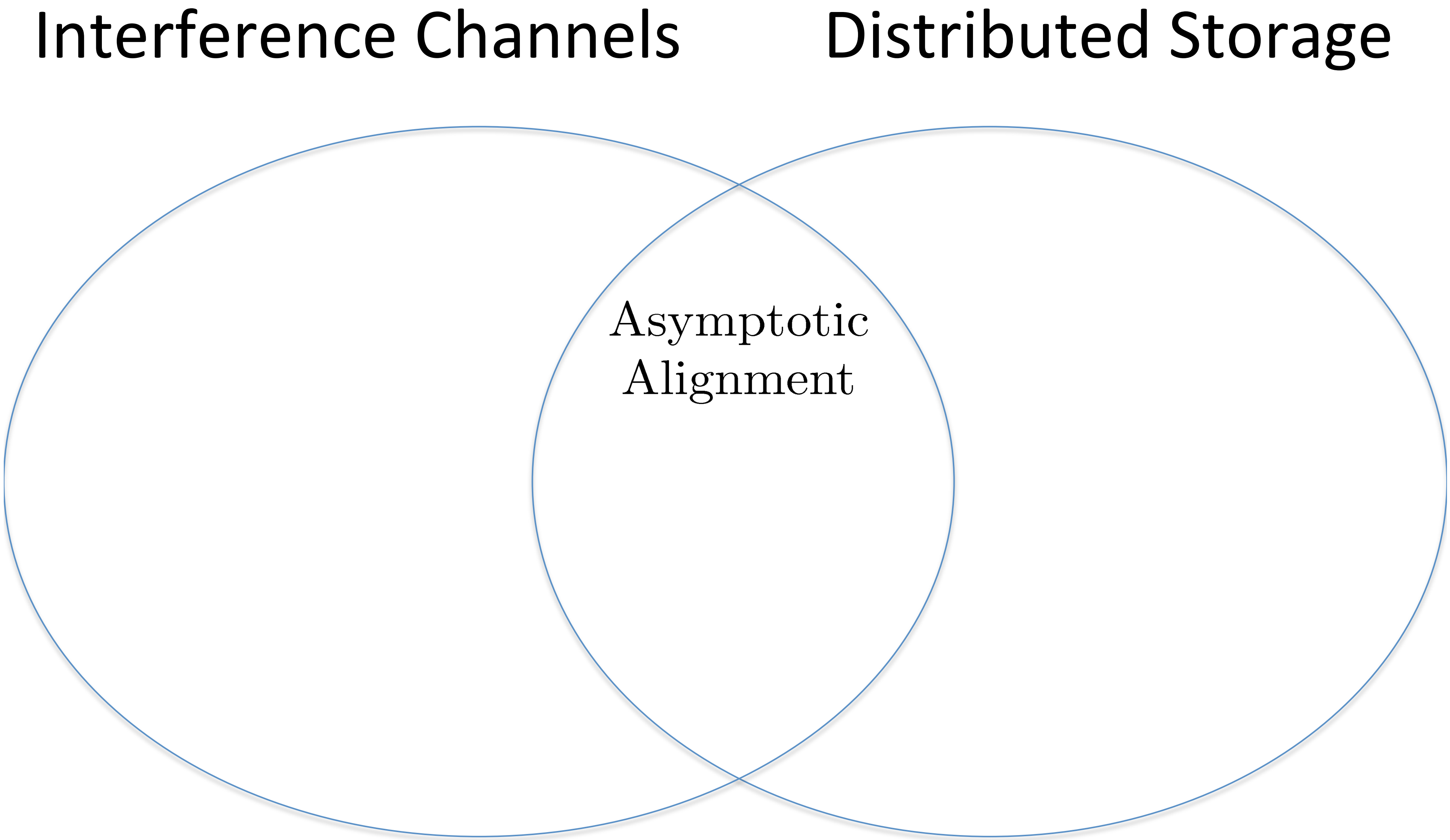


- Multi-Source (Wired) Network Capacity Problem
- Asymptotic Alignment Based Codes
  - mimic random wireless Channels for coding parities

[Cadambe-Jafar-Maleki 10, Suh-Ramchandran 10]

*For $k \leq n/2$ solved with finite alignment based codes in [Wu-Dimakis 09, Shah-Rashmi-Kumar-Ramchandran 08, Suh-Ramchandran 09]


Interference Channels
Distributed Storage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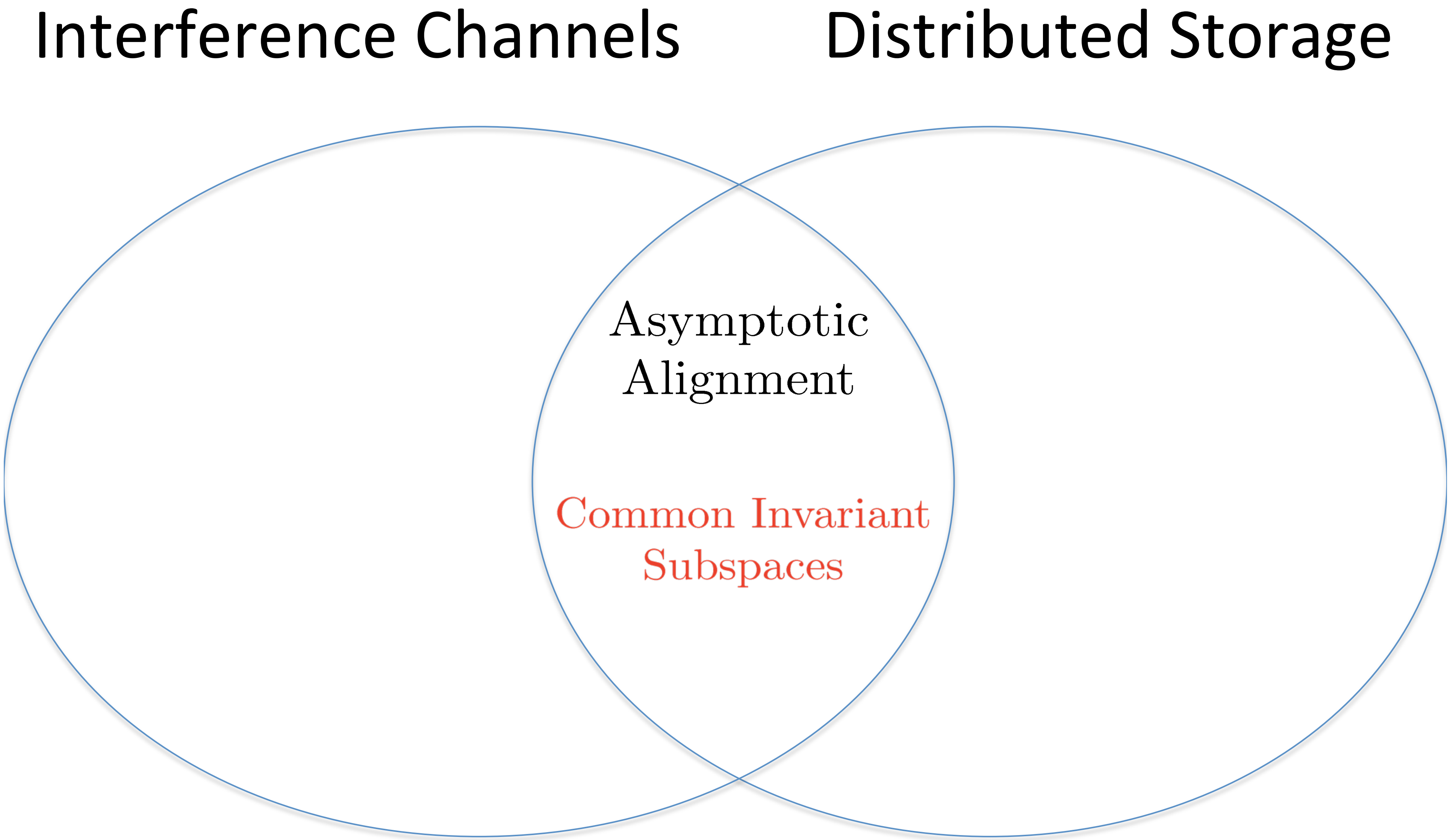
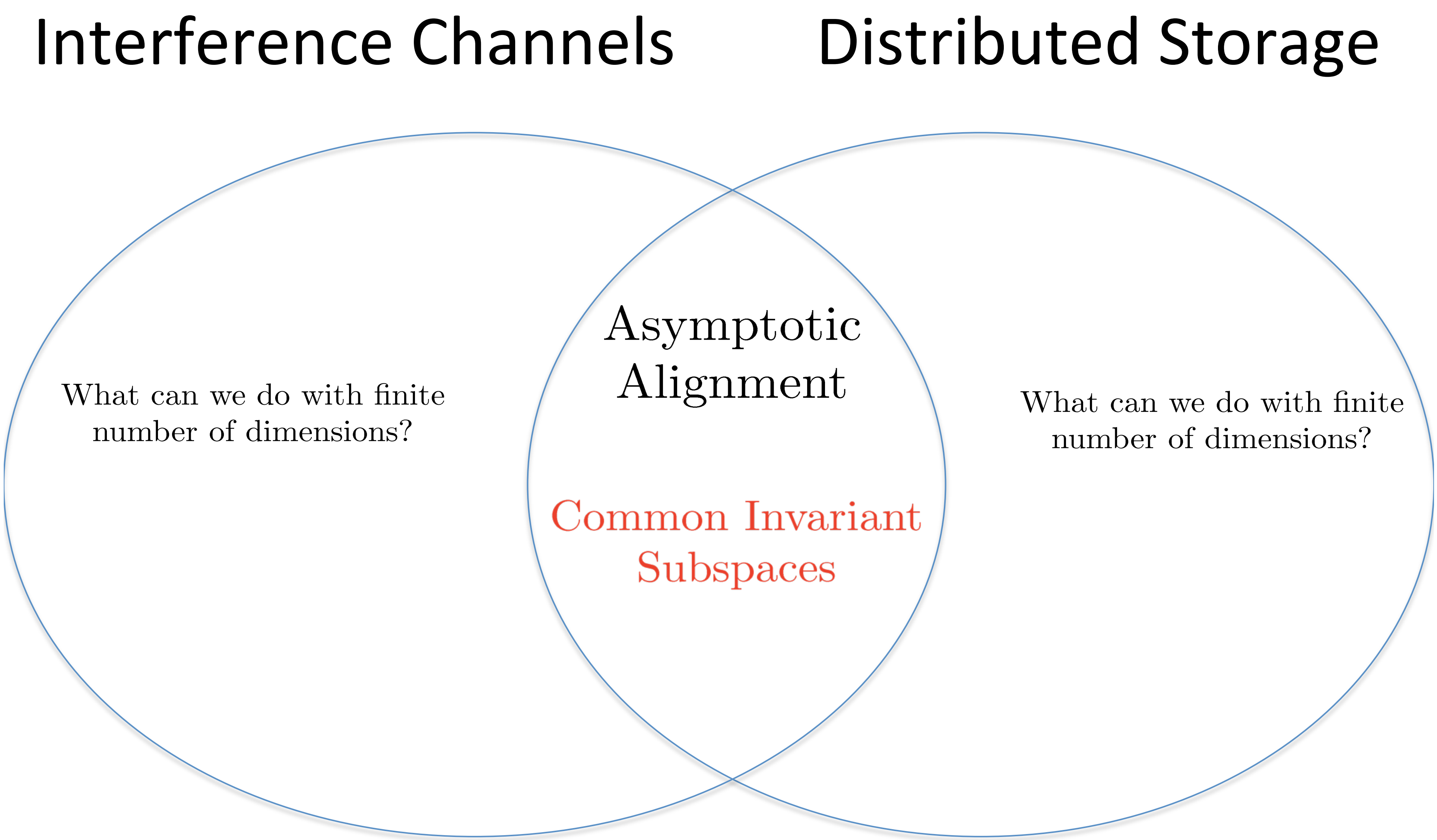
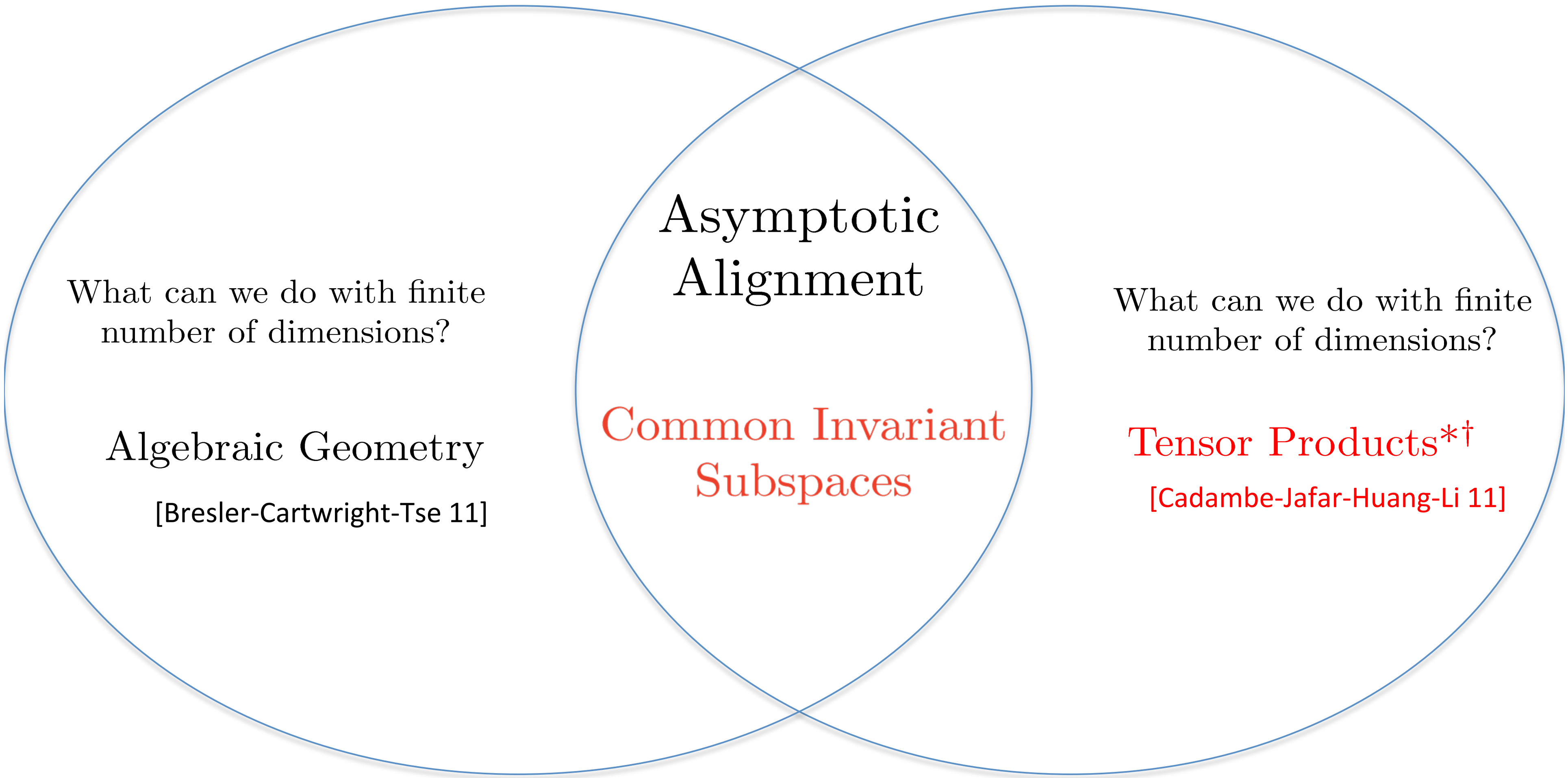

Interference Channels
Distributed Storage
Asymptotic
Alignment
Common Invariant
Subspaces



Interference Channels
Distributed Storage
What can we do with finite number of dimensions?
Asymptotic Alignment
Common Invariant Subspaces
What can we do with finite number of dimensions?

Interference Channels   Distributed Storage



† Tensor Product framework generalizes finite codes of [Cadambe-Huang-Li, Tamo-Wang-Bruck, Papailiopoulos-Dimakis ISIT 11]

*Also called *Subspace Alignment*, Introduced by [Suh-Tse 08]

Invariant Subspaces

Linear Operator (matrix) $\mathbf{T} : \mathcal{V} \to \mathcal{V}$

Subspace $\mathbf{V} \subset \mathcal{V}$ is $\mathbf{T}$-invariant iff

$$\mathrm{span}(\mathbf{TV}) \subseteq \mathrm{span}(\mathbf{V})$$

i.e., $\mathbf{V}$ *aligns with* $\mathbf{TV}$

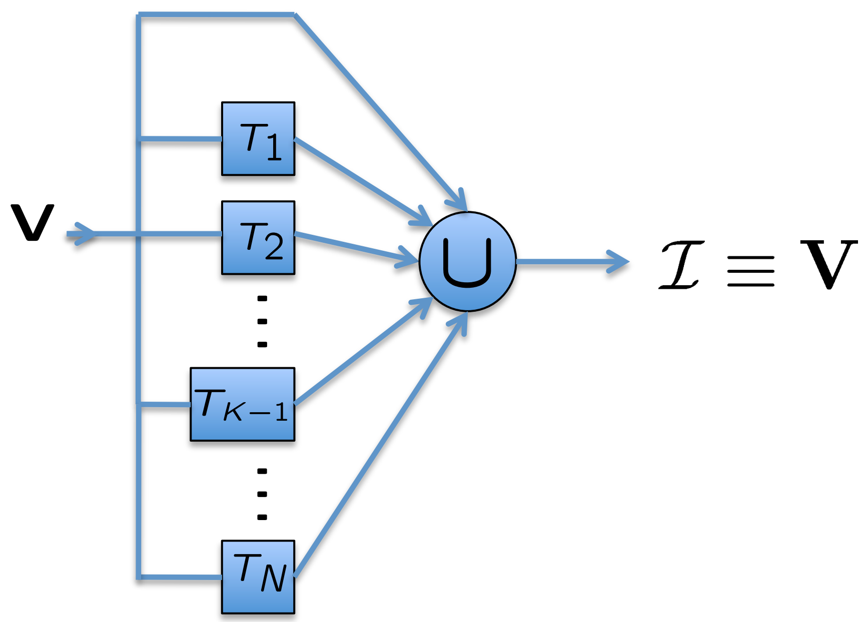
Examples:

- Trivial spaces: $\{\mathbf{0}\}$, The universal space $\mathcal{V}$.
- Eigen vector of $\mathbf{T}$

Common Invariant Subspaces

$\mathbf{V} \subseteq \mathcal{V}$ is a *Common Invariant Subspace* of $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n$ iff

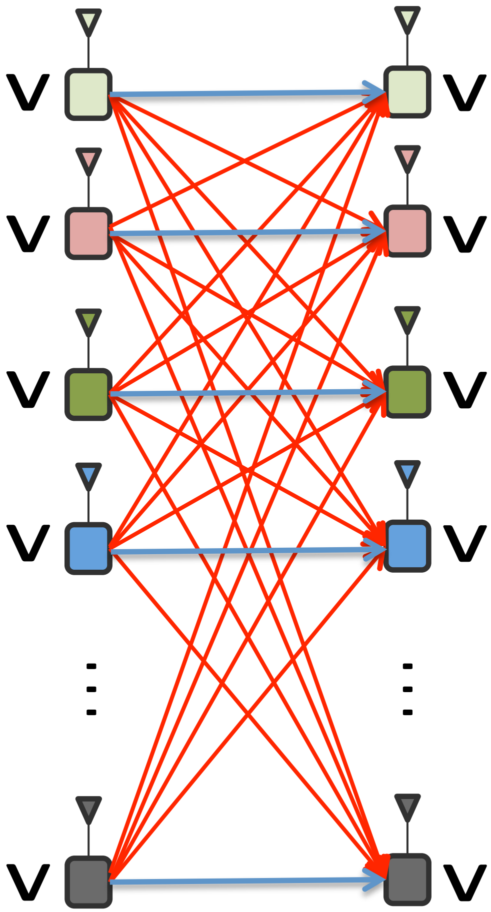
$\mathbf{V}$ is $\mathbf{T}_i$-invariant for all $i = 1, 2, \ldots, N$



Achieving $K/2$ DOF in K user Interference Channel

Achieving $K/2$ DOF in K user Interference Channel

[Cadambe, Jafar, IT08]



Ignore direct channels.

Enumerate all cross channels $T_1, T_2, \cdots, T_N$

Critical assumption
Commutative property: $T_i T_j = T_j T_i$
(e.g. time-varying/frequency selective setting $\rightarrow$ diagonal channels)

All transmitters use the *same* signal space $\mathbf{V}$

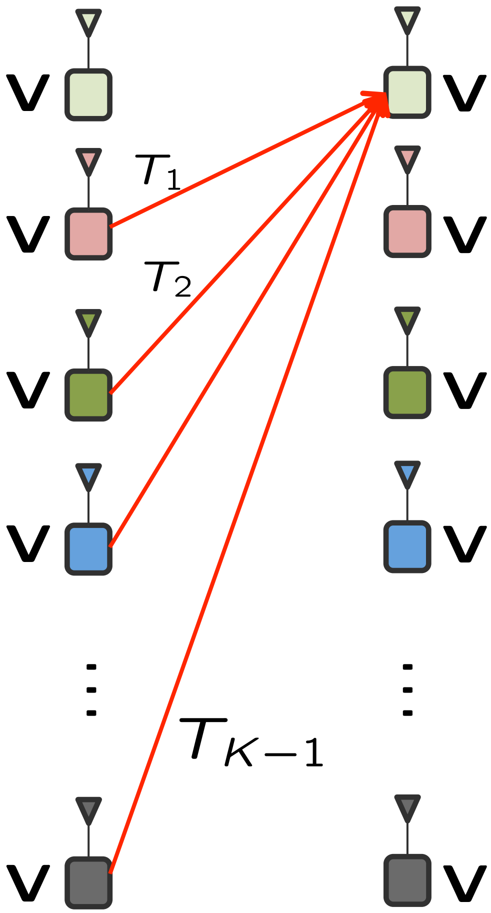
All receivers set aside the *same* interference space $\mathbf{V}$

$$\mathbf{V}^{[1]} = \mathbf{V}^{[2]} = \cdots = \mathbf{V}^{[K]} = \mathbf{V}$$

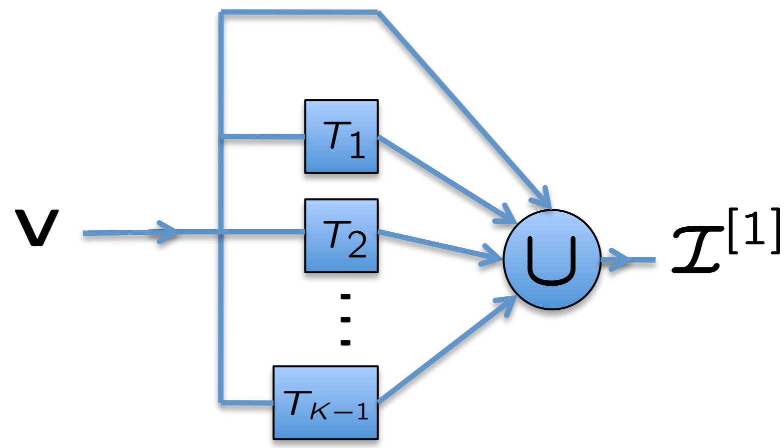
$$\mathcal{I}^{[1]} = \mathcal{I}^{[2]} = \cdots = \mathcal{I}^{[K]} = \mathbf{V}$$

Interference Alignment Scheme of [CJ08]

[Cadambe, Jafar, IT08]



What is the interference space at Receiver 1 ?



Interference Alignment Scheme of [CJ08]

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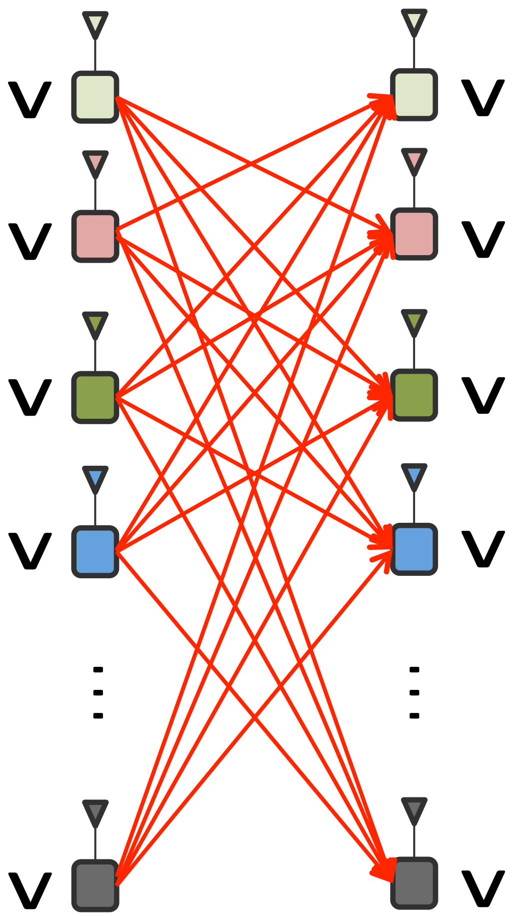
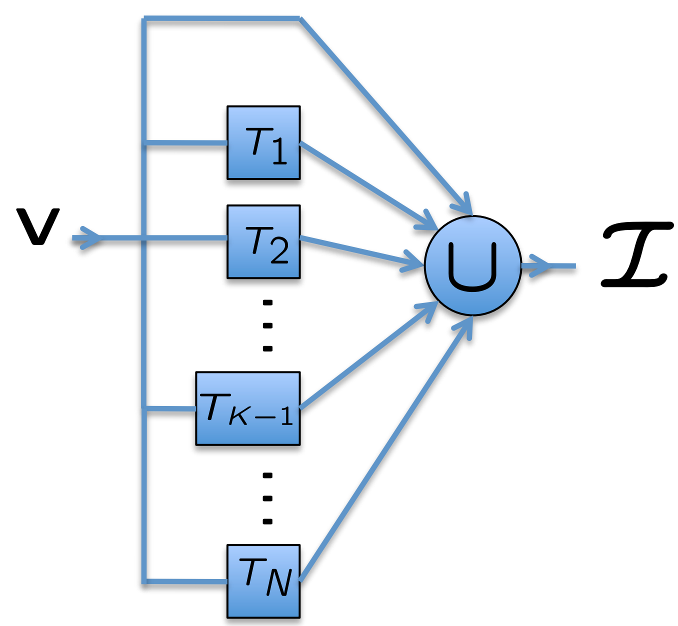


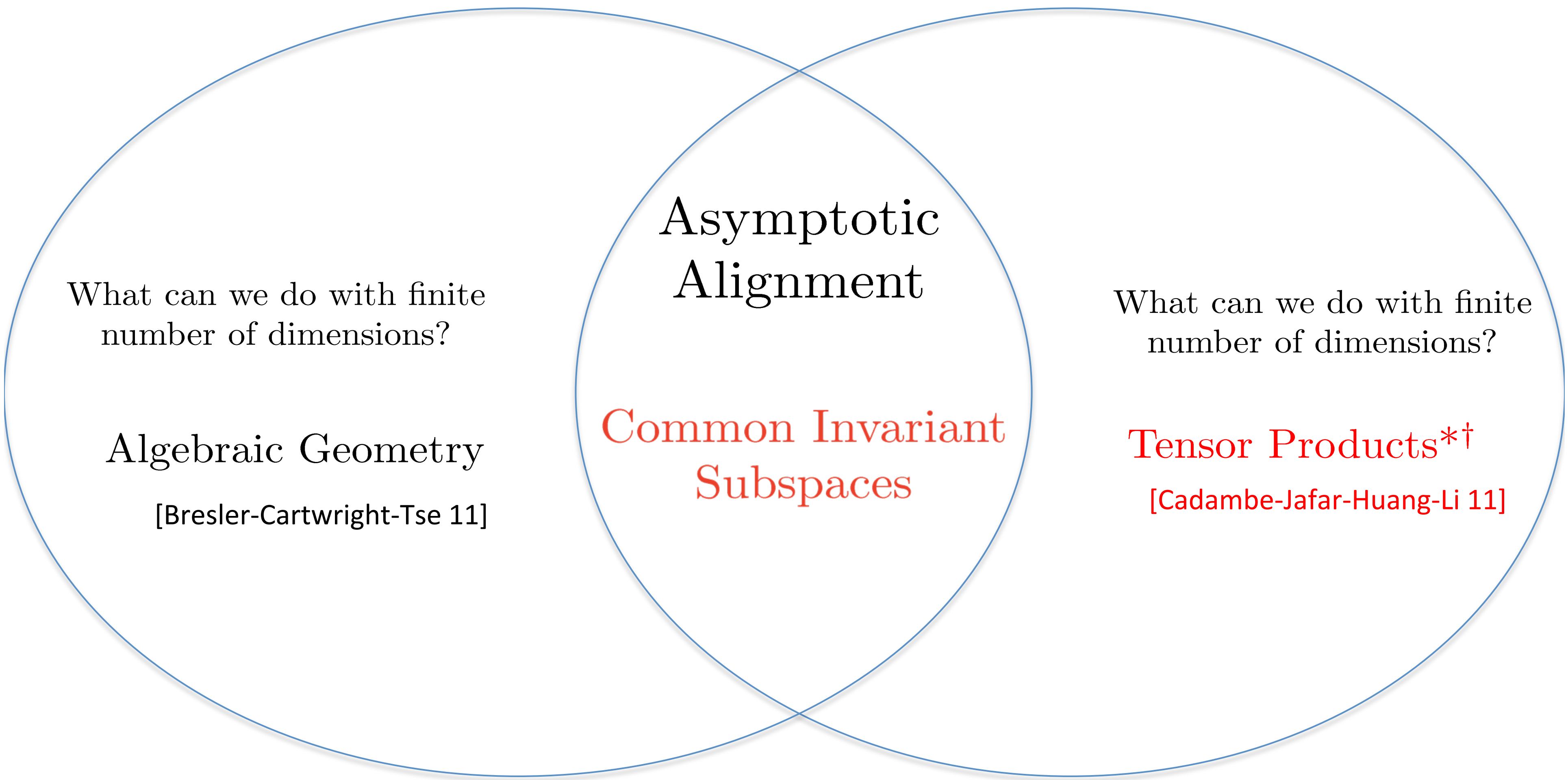
All the interference at all the receivers: $\mathcal{I}$



Goal: Make $\mathbf{V} \equiv \mathcal{I}$

Main Insight of [CJ08]: Asymptotically common invariant spaces for commutative operators.

Interference Channels   Distributed Storage



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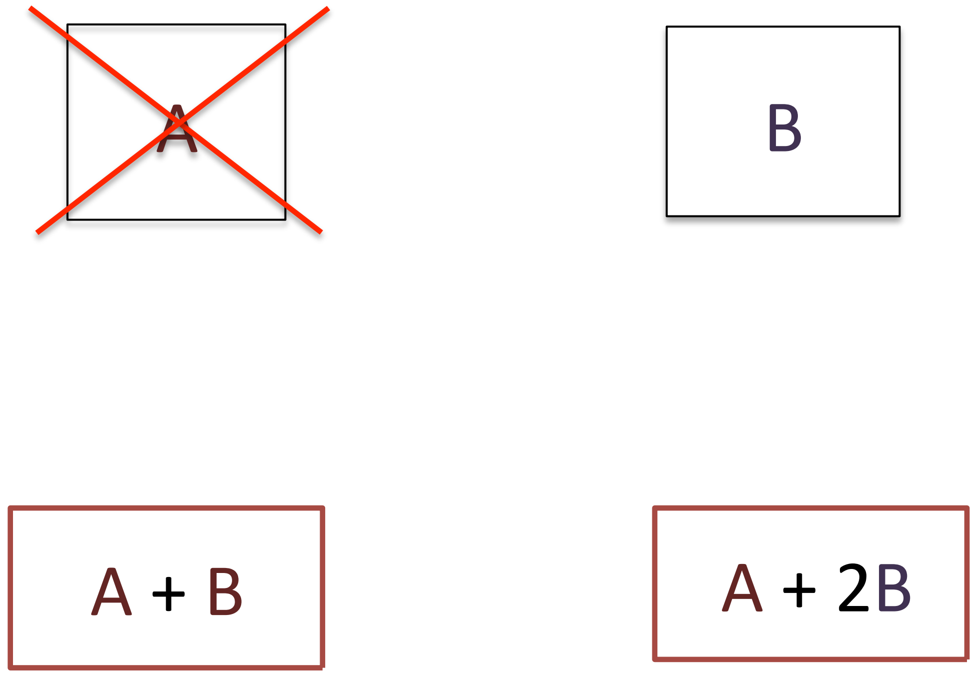
$(4,2)$ MDS Code - Repair Efficiency of a single failure



*1 unit stored in every node

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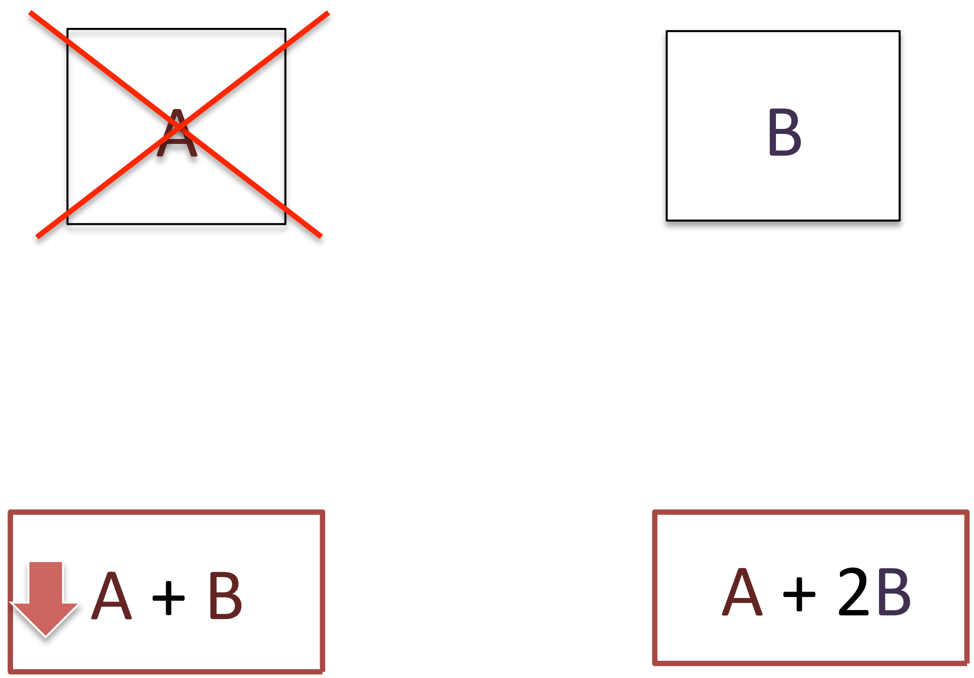

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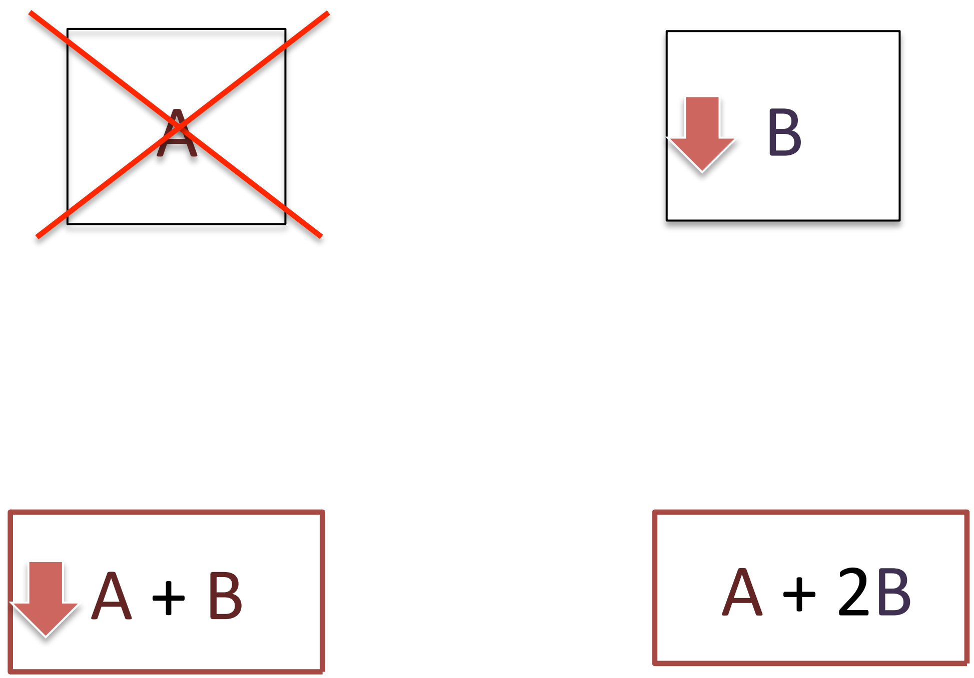

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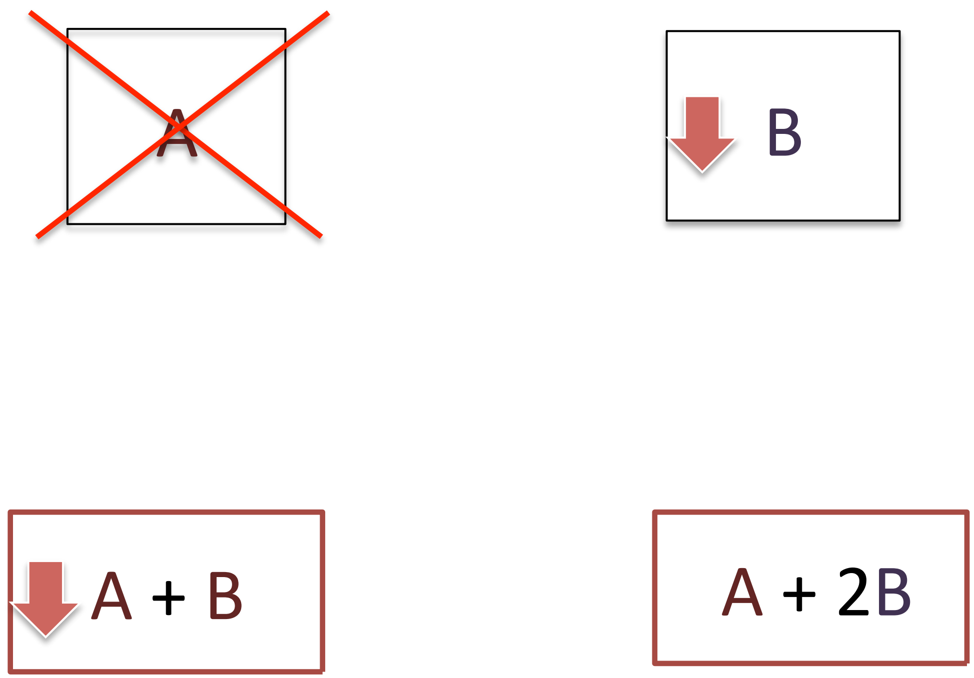
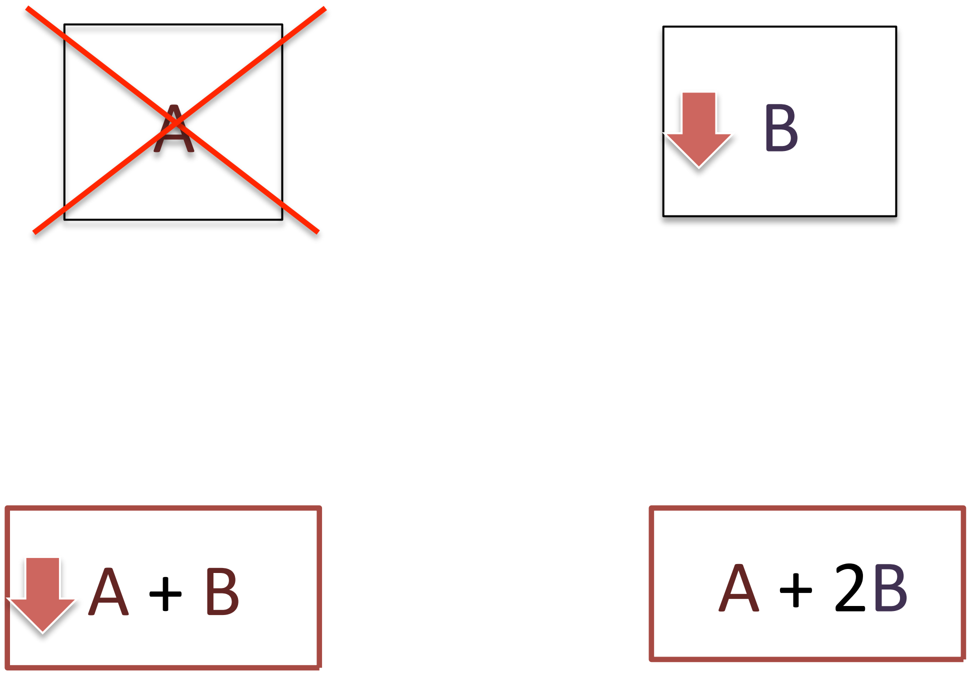


(Trivial Strategy) For a (4,2) Repair Bandwidth is 2 units

*1 unit stored in every node

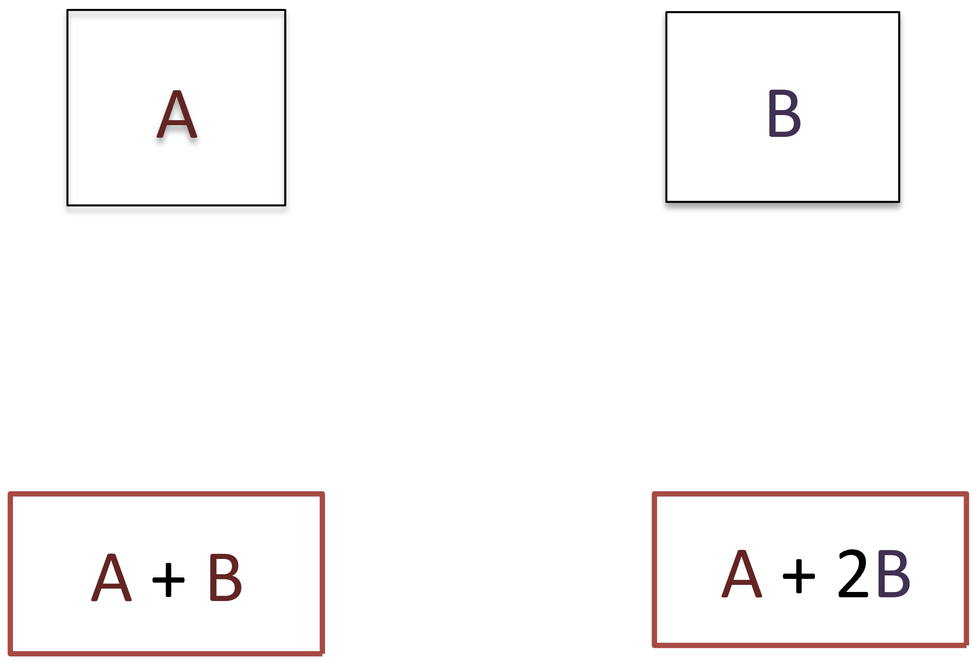
$(4,2)$ MDS Code - Repair Efficiency of a single failure



(Trivial Strategy) For a (n,k) Repair Bandwidth is k units

*1 unit stored in every node

$(4,2)$ MDS Code - Repair Efficiency of a single failure



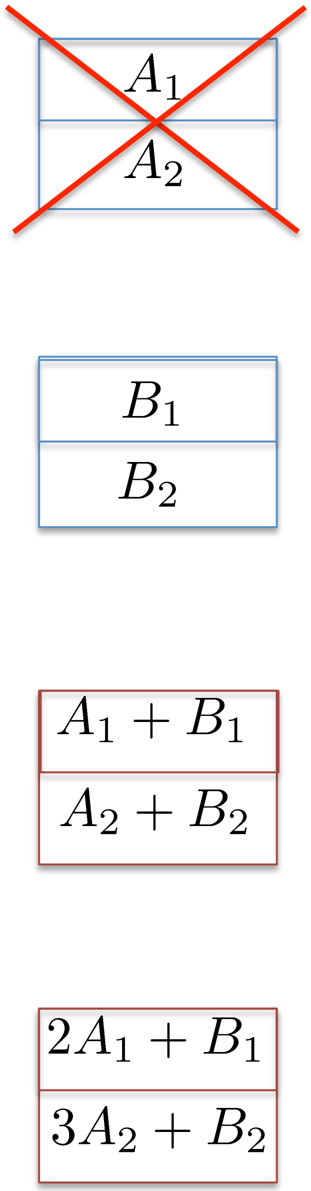
(Trivial Strategy) For a (n,k) Repair Bandwidth is k units

Can we do better?

*1 unit stored in every node

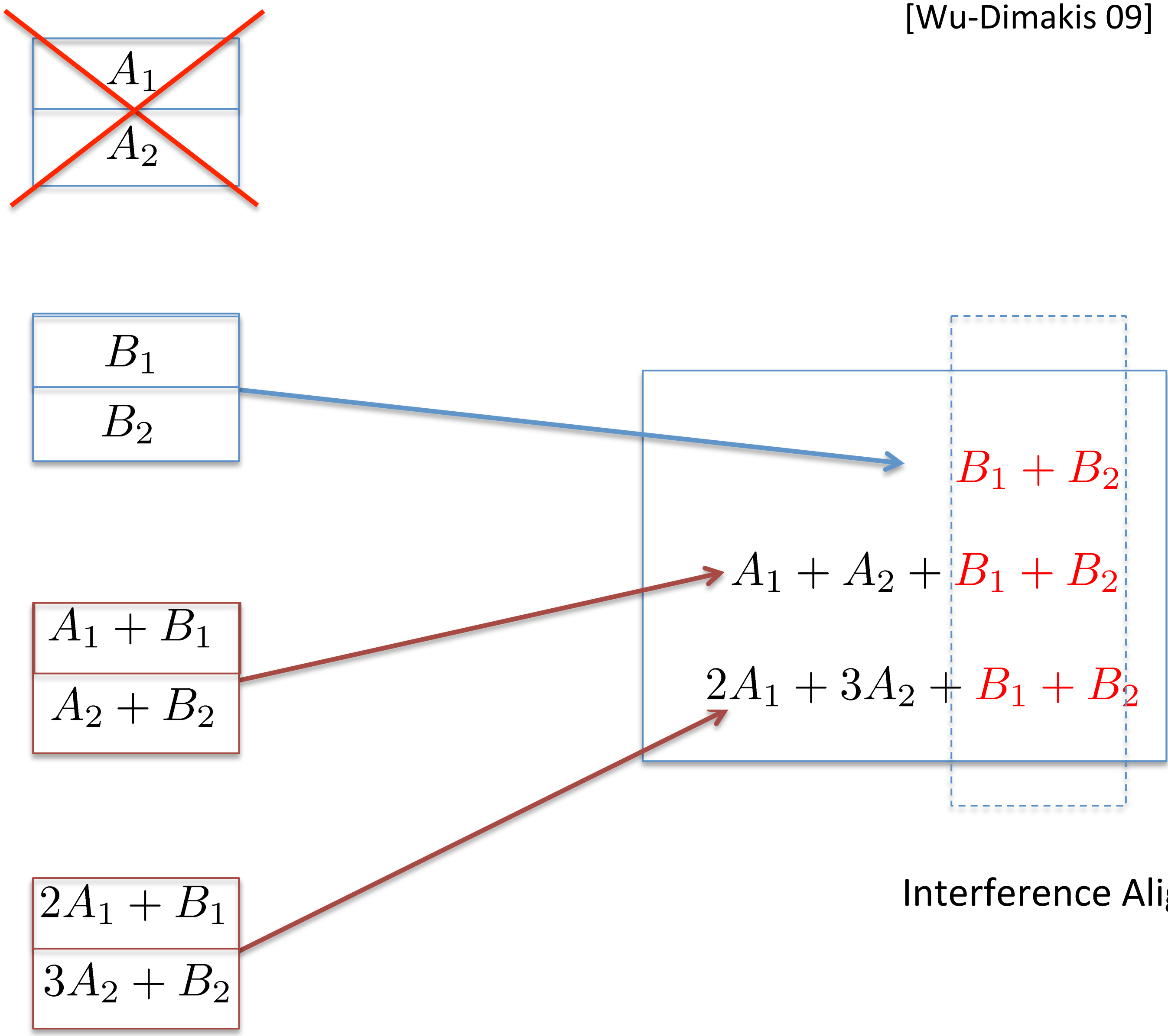
$$n = 4, k = 2$$ [Wu-Dimakis 09]

Trivial Repair : 4 linear combinations



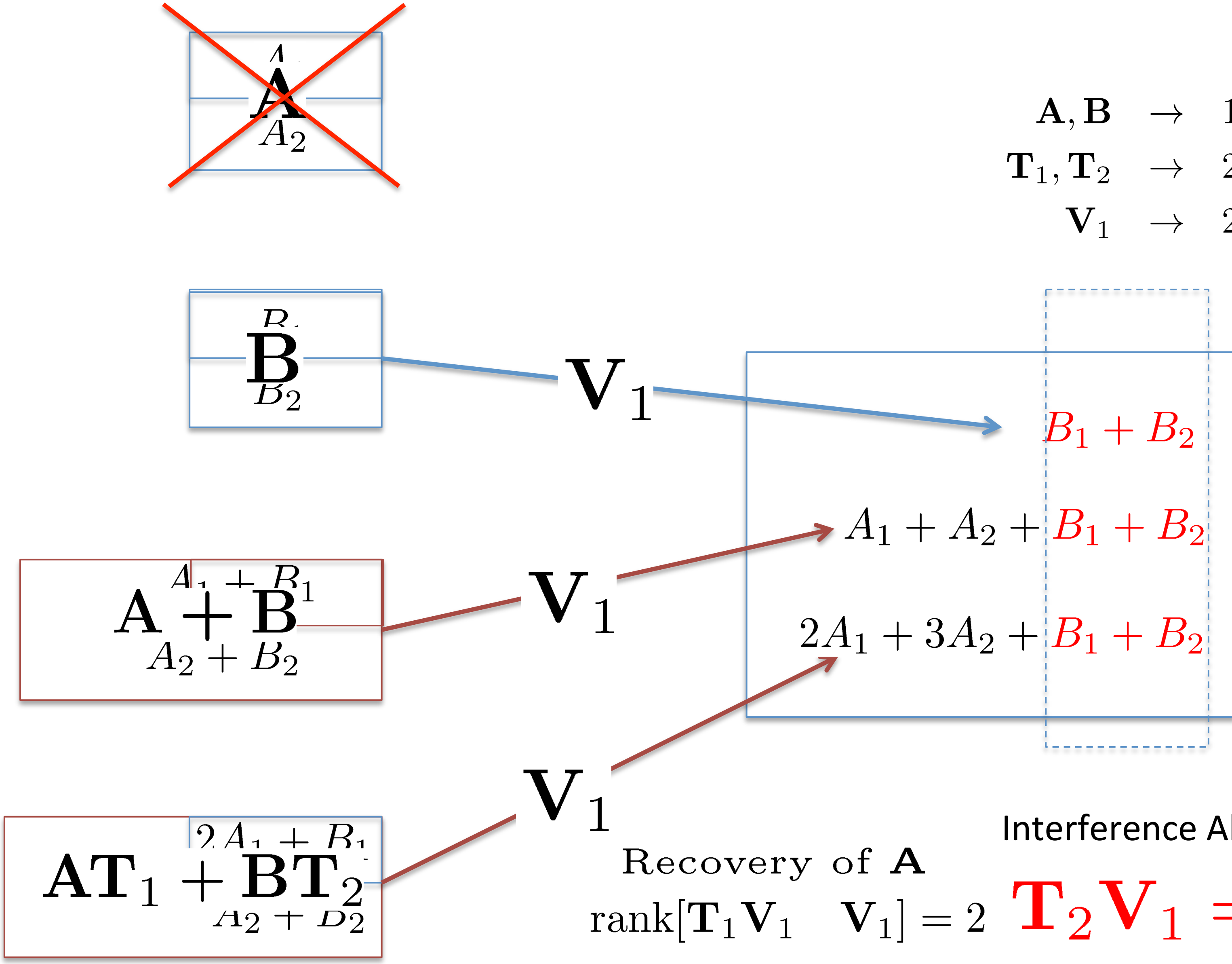
$n = 4, k = 2$, Repair with 3 Linear Combinations

[Wu-Dimakis 09]



Interference Alignment!

$n = 4, k = 2$, Repair with 3 Linear Combinations



$\mathbf{A}, \mathbf{B} \rightarrow 1 \times 2$

$\mathbf{T}_1, \mathbf{T}_2 \rightarrow 2 \times 2$

$\mathbf{V}_1 \rightarrow 2 \times 1$

Recovery of $\mathbf{A}$

$\text{rank}[\mathbf{T}_1\mathbf{V}_1 \quad \mathbf{V}_1] = 2$

Interference Alignment!

$$\mathbf{T}_2\mathbf{V}_1 = \lambda_1\mathbf{V}_1$$

Repair for n=4, k=2

$$\mathbf{T}_1, \mathbf{T}_2 \quad \rightarrow \quad 2 \times 2$$
$$\mathbf{V}_1, \mathbf{V}_2 \quad \rightarrow \quad 2 \times 1$$

Repair of Node 1

$$\mathbf{T}_2\mathbf{V}_1 = \lambda_1\mathbf{V}_1$$
$$\text{rank}[\mathbf{T}_1\mathbf{V}_1 \quad \mathbf{V}_1] = 2$$

Repair of Node 2

$$\mathbf{T}_1\mathbf{V}_2 = \lambda_2\mathbf{V}_2$$
$$\text{rank}[\mathbf{T}_2\mathbf{V}_2 \quad \mathbf{V}_2] = 2$$

$\mathbf{V}_1$ eigen-vector of $\mathbf{T}_2$
$\mathbf{V}_2$ eigen-vector of $\mathbf{T}_1$

Repair Vectors $\mathbf{V}_1, \mathbf{V}_2, \rightarrow$ Beamforming Vectors in Wireless Comm.
Coding matrices $\mathbf{T}_1, \mathbf{T}_2, \rightarrow$ Channel Matrices in Wireless Comm.
"Structured Channels"!

For optimal repair of (k+2, k) codes, we need

$\mathbf{T}_i$ full rank $M \times M$
$\mathbf{V}$ is $M \times M/2$

$$\text{span}(\mathbf{T}_i \mathbf{V}_j) = \text{span}(\mathbf{V}_j), i \neq j, i, j = 1, 2, \ldots, k$$

$$\text{span}(\mathbf{T}_i \mathbf{V}_i) \cap \text{span}(\mathbf{V}_i) = \{0\}$$

Solution 1: [Cadambe-Jafar-Maleki 10, Suh-Ramchandran 10]

- Choose $\mathbf{T}_i, i = 1, 2 \ldots, k$ random diagonal
- Choose $\mathbf{V}_i, i = 1, 2, \ldots, k$ according to asymptotic alignment

Solution 2: Next

For optimal repair of (5,3) code

$\mathbf{T}_i$ full rank $M \times M$
$\mathbf{V}$ is $M \times M/2$
$M = 8$

$$\mathrm{span}(\mathbf{T}_2\mathbf{V}_1) = \mathrm{span}(\mathbf{T}_3\mathbf{V}_1) = \mathrm{span}(\mathbf{V}_1)$$

$$\mathrm{span}(\mathbf{T}_1\mathbf{V}_1) \cap \mathrm{span}(\mathbf{V}_1) = \{0\}$$

$$\mathrm{span}(\mathbf{T}_1\mathbf{V}_2) = \mathrm{span}(\mathbf{T}_3\mathbf{V}_2) = \mathrm{span}(\mathbf{V}_2)$$

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What are Tensor Pr$\otimes$ducts?

Tensor Product of (multiple) vectors is a vector, (multiple) vector spaces is a vector space, (multiple) transformations is a transformation .

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Let $\mathcal{B} = \{b_1, b_2\}$ be a basis of (2 dimensional) vector space $\mathcal{V}$

Then $\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}$ is a vector space with basis $\mathcal{B} \times \mathcal{B} \times \mathcal{B}$

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Then, extend by multi-Linearity (Key Property)

$$(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) \otimes \mathbf{v} \otimes \mathbf{w} = \alpha(\mathbf{u}_1 \otimes \mathbf{v} \otimes w) + \beta(\mathbf{u}_2 \otimes \mathbf{v} \otimes \mathbf{w})$$

etc.

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etc.

Example:

$$(b_1 - b_2) \otimes (b_1 + b_2) \otimes b_1 = (b_1 \otimes b_1 \otimes b_1) + (b_1 \otimes b_2 \otimes b_1) - (b_2 \otimes b_1 \otimes b_1) - (b_2 \otimes b_2 \otimes b_1)$$

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etc.

Alignment properties of tensor products

Alignment of each factor of the product
ensures alignment of the product

$$\mathrm{span}(\mathbf{V}_i) = \mathrm{span}(\mathbf{U}_i) \Rightarrow \mathrm{span}(\mathbf{V}_1 \otimes \mathbf{V}_2 \otimes \mathbf{V}_3) = \mathrm{span}(\mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3)$$

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“Non-alignment” of at least one factor of the product
ensures “non-alignment” of the product

$$\mathrm{span}(\mathbf{V}_1) \cap \mathrm{span}(\mathbf{U}_1) = \{0\}$$

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Mixed Product Property

$$(\mathbf{H}_1 \otimes \mathbf{H}_2 \otimes \mathbf{H}_3)(\mathbf{V}_1 \otimes \mathbf{V}_2 \otimes \mathbf{V}_3) = (\mathbf{H}_1\mathbf{V}_1 \otimes \mathbf{H}_2\mathbf{V}_2 \otimes \mathbf{H}_3\mathbf{V}_3)$$

Let

$$\mathbf{H} = \mathbf{H}_1 \otimes \mathbf{H}_2 \otimes \mathbf{H}_3$$

$$\mathbf{U} = \mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3$$

Let

$$\mathbf{H} = \mathbf{H}_1 \otimes \mathbf{H}_2 \otimes \mathbf{H}_3$$

$$\mathbf{U} = \mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3$$

$\mathbf{U}$ is $\mathbf{H}$-invariant iff
$\mathbf{U}_1$ is $\mathbf{H}_1$ invariant *and*
$\mathbf{U}_2$ is $\mathbf{H}_2$ invariant *and*
$\mathbf{U}_3$ is $\mathbf{H}_3$ invariant.

$\text{span}(\mathbf{U}) \cap \text{span}(\mathbf{HU}) = \{0\}$ if
$\text{span}(\mathbf{U}_1) \cap \text{span}(\mathbf{H}_1\mathbf{U}_1) = \{0\}$ or
$\text{span}(\mathbf{U}_2) \cap \text{span}(\mathbf{H}_2\mathbf{U}_2) = \{0\}$ or
$\text{span}(\mathbf{U}_3) \cap \text{span}(\mathbf{H}_3\mathbf{U}_3) = \{0\}$

For optimal repair of (5,3) code

$\mathbf{T}_i$ full rank $M \times M$
$\mathbf{V}$ is $M \times M/2$
$M = 8$

$$\operatorname{span}(\mathbf{T}_2\mathbf{V}_1) = \operatorname{span}(\mathbf{T}_3\mathbf{V}_1) = \operatorname{span}(\mathbf{V}_1)$$
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$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \mathbf{I}_2 \otimes \mathbf{I}_2$$

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$$\mathbf{T}_2 = \mathbf{I}_2 \otimes \; ? \otimes ?$$

$$\mathbf{T}_3 = \mathbf{I}_2 \otimes \; ? \otimes ?$$

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \mathbf{I}_2 \otimes \mathbf{I}_2$$

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$M = 8$

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$$\text{span}(\mathbf{T}_1\mathbf{V}_1) \cap \text{span}(\mathbf{V}_1) = \{0\}$$

Invariance

$$\mathbf{T}_2 = \mathbf{I}_2 \otimes \; ? \; \otimes ?$$

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \mathbf{I}_2 \otimes \mathbf{I}_2$$

$$\mathbf{T}_3 = \mathbf{I}_2 \otimes \; ? \; \otimes ?$$

? any full rank matrix

For optimal repair of (5,3) code,

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$$\mathbf{T}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes ?? \otimes ??$$

Distinguishability

$$\mathbf{T}_3 = \mathbf{I}_2 \otimes ? \otimes ?$$

? any full rank matrix

$$\mathbf{T}_1 = \lambda_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{I}_2 \otimes \mathbf{I}_2 \qquad \mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \mathbf{I}_2 \otimes \mathbf{I}_2$$

$$\mathbf{T}_2 = \lambda_2 \mathbf{I}_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{I}_2 \qquad \mathbf{V}_2 = \mathbf{I}_2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \mathbf{I}_2$$

$$\mathbf{T}_3 = \lambda_3 \mathbf{I}_2 \otimes \mathbf{I}_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{V}_3 = \mathbf{I}_2 \otimes \mathbf{I}_2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Can use Ergodic Alignment matrices or other classes of matrices
- For $(k+2, k)$ codes, use Tensor product of k two-dimensional spaces

Speculations? Musings? Open Problems?

- For DOF characterizations, no unified technique exists - even when restricted to linear (beamforming) schemes. Are (superposition of) Common Invariant Subspaces fundamental structures?
  - Note: Distributed Lattices aligning over different channel gains are *common invariant subgroups*
- Distributed Storage: Is Asymptotics alignment *necessary* in general?
  - For multiple node failures, connection to arbitrary subsets of surviving nodes....

Extra Slides

Multi-Source Network