Tail inequalities for order statistics of log-concave vectors and applications

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Basic definitions and Notation

Let $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ with full dimensional support. We say that the distribution of X is

- *logaritmically concave*, if X has density of the form $e^{-h(x)}$ with $h\colon \mathbb{R}^n \to (-\infty, \infty]$ convex;
- *isotropic*, if $\mathbb{E}X_i = 0$ and $\mathbb{E}X_iX_j = \delta_{i,j}$.

For $x \in \mathbb{R}^n$ we put

- $|x| = \|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$
- $\|x\|_r = \left(\sum_{i=1}^n |x_i|^r\right)^{1/r}$, $1 \le r < \infty$, $\|x\|_\infty = \max_i |x_i|$
- $P_I x$ - canonical projection of x onto $\{y \in \mathbb{R}^n\colon \ \mathrm{supp}(y) \subset I\}$, $\emptyset \neq I \subset \{1, \ldots, n\}$.

Order Statistics

For an n–dimensional random vector X by $X_1^* \geq X_2^* \geq \ldots \geq X_n^*$ we denote the nonincreasing rearrangement of $|X_1|, \ldots, |X_n|$ (in particular $X_1^* = \max\{|X_1|, \ldots, |X_n|\}$ and $X_n^* = \min\{|X_1|, \ldots, |X_n|\}$). Random variables X_k^*, $1 \leq k \leq n$, are called order statistics of X.

Problem Find upper bound for $\mathbb{P}(X_k^* \geq t)$.

If coordinates of X_i are independent symmetric exponential r.v. with variance 1 then $\mathrm{Med}(X_k^*) \sim \log(en/k)$ for $k \leq n/2$.

Union bound

We have for isotropic logconcave vectors X,

$$\begin{aligned}
\mathbb{P}(X_k^* \geq t) &= \mathbb{P}\Big(\bigcup_{i_1<\ldots<i_k} \{|X_{i_1}| \geq t, \ldots, |X_{i_k}| \geq t\}\Big) \\
&\leq \sum_{i_1<\ldots<i_k} \sum_{\eta_1=\pm 1,\ldots,\eta_k=\pm 1} \mathbb{P}(\eta_1 X_{i_1} \geq t, \ldots, \eta_k X_{i_k} \geq t) \\
&\leq \sum_{i_1<\ldots<i_k} \sum_{\eta_1=\pm 1,\ldots,\eta_k=\pm 1} \mathbb{P}\Big(\frac{1}{\sqrt{k}}(\eta_1 X_{i_1} + \ldots + \eta_k X_{i_k}) \geq t\sqrt{k}\Big) \\
&\leq \binom{n}{k} 2^k \exp\Big(-\frac{1}{C} t\sqrt{k}\Big).
\end{aligned}$$

Therefore

$$\mathbb{P}(X_k^* \geq t) \leq \exp\Big(-\frac{1}{C} t\sqrt{k}\Big) \quad \text{for } t \geq C\sqrt{k}\log\Big(\frac{en}{k}\Big).$$

Exponential concentration

Random vector X in $\mathbb{R}^n$ satisfies *exponential concentration inequality with* a constant α if

$$\mathbb{P}(X \in A + \alpha t B_2^n) \geq 1 - \exp(-t) \quad \text{if } \mathbb{P}(X \in A) \geq \frac{1}{2} \text{ and } t > 0.$$

Conjecture (Kannan-Lovasz-Simonovits)

Isotropic log-concave vectors satisfy exponential concentration with universal α

Known to hold for unconditional permutationally invariant isotropic log-concave vectors (Klartag'11+).

The best known bound for general case is $\alpha \leq n^{5/12}$ (Guedon-Milman'10+).

Order statistics under exponential concentration

Proposition

If X is isotropic n-dimensional and satisfies exponential concentration inequality with a constant $\alpha \geq 1$ then

$$\mathbb{P}(X_k^* \geq t) \leq \exp\Big(-\frac{1}{3\alpha}\sqrt{k}t\Big) \quad \textit{for } t \geq 8\alpha\log\Big(\frac{en}{k}\Big).$$

Sketch of the proof. The set

$$A := \Big\{x \in \mathbb{R}^n\colon\ \#\Big\{i\colon |x_i| \geq 4\alpha\log\Big(\frac{en}{k}\Big)\Big\} < \frac{k}{2}\Big\}.$$

has measure μ at least $1/2$. If $z = x + y \in A + \sqrt{k}sB_2^n$ then less than $k/2$ of $|x_i|$'s are bigger than $4\alpha\log(en/k)$ and less than $k/2$ of $|y_i|$'s are bigger than $\sqrt{2}s$, so

$$\mathbb{P}\Big(X_k^* \geq 4\alpha\log\Big(\frac{en}{k}\Big) + \sqrt{2}s\Big) \leq 1 - \mu(A + \sqrt{k}sB_2^n) \leq \exp(-\frac{1}{\alpha}\sqrt{k}s).$$

Order Statistics for isotropic log-concave vectors

Kannan-Lovasz-Simonovits Conjecture is open, nevertheless one may show the estimate for order statistics.

Theorem

Let X be n-dimensional log-concave isotropic vector. Then

$$\mathbb{P}(X_k^* \geq t) \leq \exp\Big(-\frac{1}{C}\sqrt{k}t\Big) \quad \text{for } t \geq C\log\Big(\frac{en}{k}\Big).$$

Our approach is based on the suitable estimate of moments of the process $N_X(t)$, where

$$N_X(t) := \sum_{i=1}^{n} \mathbb{1}_{\{X_i \geq t\}} \quad t \geq 0.$$

Estimate for N_X

Theorem

For any isotropic log-concave vector X and $p \geq 1$ we have

$$\mathbb{E}(t^2 N_X(t))^p \leq (Cp)^{2p} \quad \text{for } t \geq C\log\Big(\frac{nt^2}{p^2}\Big).$$

To get estimate for order statistics we observe that $X_k^* \geq t$ implies that $N_X(t) \geq k/2$ or $N_{-X}(t) \geq k/2$ and vector $-X$ is also isotropic and log-concave. Estimates for N_X and Chebyshev's inequality gives

$$\mathbb{P}(X_k^* \geq t) \leq \Big(\frac{2}{k}\Big)^p (\mathbb{E}N_X(t)^p + \mathbb{E}N_{-X}(t)^p) \leq 2\Big(\frac{Cp}{t\sqrt{k}}\Big)^{2p}$$

provided that $t \geq C\log(nt^2/p^2)$. We take $p = \frac{1}{eC}t\sqrt{k}$ and notice that the restriction on t follows by the assumption that $t \geq C\log(en/k)$.

Paouris Theorem

Proof of estimate for $N_X(t)$ is based on two ideas. First is that the restriction of a log-concave vector X to a convex set is log-concave. Second is Paouris' concentration of mass result.

Theorem (Paouris)

For any isotropic log-concave vector X in $\mathbb{R}^n$,

$$\mathbb{P}(|X| \geq t) \leq \exp\left(-\frac{1}{C}t\right) \quad \textit{for } t \geq C\sqrt{n},$$

equivalently

$$(\mathbb{E}|X|^p)^{1/p} \leq C\left(\sqrt{n} + p\right) \quad \textit{for } p \geq 2.$$

Estimate for N_X implies Paouris concentration

Proposition

Suppose that X is a random vector in $\mathbb{R}^n$ such that

$$\mathbb{E}(t^2 N_{UX}(t))^l \le (A_1 l)^{2l} \quad \text{for } t \ge A_2,\ l \ge \sqrt{n},\ U \in O(n),$$

where A_1, A_2 are finite constants. Then

$$\mathbb{P}(|X| \ge t\sqrt{n}) \le \exp\Big(-\frac{1}{CA_1} t\sqrt{n}\Big) \quad \text{for } t \ge \max\{CA_1, A_2\}.$$

Idea of the proof. For any $U_1, \ldots, U_l \in O(n)$,

$$\mathbb{E}\prod_{i=1}^{l} N_{U_i X}(t) \le \Big(\prod_{i=1}^{l} \mathbb{E} N_{U_i X}(t)^l\Big)^{1/l} \le \Big(\frac{A_1 l}{t}\Big)^{2l} \quad \text{for } l \ge \sqrt{n}.$$

If $U_1, \ldots, U_l$ are random rotations then one may show that

$$\mathbb{E}_X \mathbb{E}_U \prod_{i=1}^{l} N_{U_i X}(t) = \mathbb{E}_X(\mathbb{E}_{U_1} N_{U_1 X}(t))^l \ge n^l C^{-l} \mathbb{P}(|X| \ge 2t\sqrt{n})$$

and we take $l = \lceil \sqrt{n} t / (\sqrt{eC_1} A_1) \rceil$.

Concentration of l_r norms, $1 \le r < 2$

Problem. What is the concentration for l_r norms of X?

Case $1 \le r \le 2$ reduces to the Paouris result for $r = 2$, since by the Hölder's inequality $\|X\|_r \le n^{1/r-1/2}|X|$. Thus

$$(\mathbb{E}\|X\|_r^p)^{1/p} \le C(n^{1/r} + n^{1/r-1/2}p)$$

and

$$\mathbb{P}(\|X\|_r \ge t) \le \exp\Big(-\frac{1}{C}tn^{1/2-1/r}\Big) \quad \text{for } t \ge Cn^{1/r}.$$

These bounds are optimal.

Concentration of l_r norms, $r > 2$

Example It is not hard to see that if $X_1, \ldots, X_n$ are independent symmetric exponential r.v.'s with variance one then

$$(\mathbb{E}\|X\|_r^p)^{1/p} \geq \frac{1}{C}(rn^{1/r} + p) \quad \text{for } p \geq 2, r \geq 2, n \geq C^r.$$

Theorem

For any $\delta > 0$ there exist constants $C_1(\delta), C_2(\delta) \leq C\delta^{-1/2}$ such that for any isotropic logconcave vector X and $r \geq 2 + \delta$,

$$(\mathbb{E}\|X\|_r^p)^{1/p} \leq C_2(\delta)\Big(rn^{1/r} + p\Big) \quad \text{for } p \geq 2.$$

Equivalently

$$\mathbb{P}(\|X\|_r \geq t) \leq \exp\Big(-\frac{1}{C_1(\delta)}t\Big) \quad \text{for } t \geq C_1(\delta)rn^{1/r}.$$

Concentration of l_r norms, $r > 2$ - idea of the proof

We have

$$\|X\|_r = \Big(\sum_{i=1}^n |X_i|^r\Big)^{1/r} = \Big(\sum_{i=1}^n |X_i^*|^r\Big)^{1/r} \le \Big(2\sum_{k=0}^{s-1} 2^k |X_{2^k}^*|^r\Big)^{1/r},$$

where $s = \lfloor \log_2 n \rfloor$. It is easy to check that

$$\sum_{k=0}^{s} 2^k \log^r(en2^{-k}) \le Cn \sum_{j=1}^{\infty} j^r 2^{-j} \le (Cr)^r n.$$

Thus for $t_1, \ldots, t_k \ge 0$ we get

$$\mathbb{P}\Big(\|X\|_r \ge C\Big(rn^{1/r} + \Big(\sum_{k=0}^{s} t_k\Big)^{1/r}\Big)\Big)$$

$$\le \mathbb{P}\Big(\sum_{k=0}^{s} 2^k(|X_{2^k}^*|^r - C_3^r \log^r(en2^{-k})) \ge \sum_{k=0}^{s} t_k\Big)$$

$$\le \sum_{k=0}^{s} \exp\Big(-\frac{1}{C} 2^{\frac{k}{2}-\frac{k}{r}} t_k^{1/r}\Big).$$

Uniform Paouris-type estimate

Theorem

For any $m \leq n$ and any isotropic log-concave vector X in $\mathbb{R}^n$ we have for $t \geq 1$,

$$\mathbb{P}\Big(\sup_{|I|=m} |P_I X| \geq Ct\sqrt{m}\log\Big(\frac{en}{m}\Big)\Big) \leq \exp\Big(-\frac{t\sqrt{m}}{\sqrt{\log(em)}}\log\Big(\frac{en}{m}\Big)\Big).$$

Idea of the proof. We have

$$\sup_{\substack{I\subseteq\{1,\ldots,N\}\\ |I|=m}} |P_I X| = \Big(\sum_{k=1}^{m} |X_k^*|^2\Big)^{1/2} \leq 2\Big(\sum_{i=0}^{s-1} 2^i |X_{2^i}^*|^2\Big)^{1/2},$$

where $s = \lceil \log_2 m \rceil$.

Weak parameter

For a vector X in $\mathbb{R}^n$ we define

$$\sigma_X(p) := \sup_{t \in S^{n-1}} (\mathbb{E}|\langle t, X\rangle|^p)^{1/p} \quad p \geq 2.$$

Examples

- For isotropic log-concave vectors X, $\sigma_X(p) \leq p/\sqrt{2}$.
- For subgaussian vectors X, $\sigma_X(p) \leq C\sqrt{p}$.
- We say that an isotropic vector X is ψ_α if $\sigma_X(p) \leq Cp^{1/\alpha}$ (uniform distributions on suitable normalized B_r^n balls are ψ_α with $\alpha = \min(r, 2)$)

Paouris theorem with weak parameter

Theorem (Paoouris)

For any log-concave random vector X,

$$(\mathbb{E}|X|^p)^{1/p} \leq C\Big((\mathbb{E}|X|^2)^{1/2} + \sigma_X(p)\Big) \quad \text{for } p \geq 2,$$

$$\mathbb{P}(|X| \geq t) \leq \exp\Big(-\sigma_X^{-1}\Big(\frac{t}{C}\Big)\Big) \quad \text{for } t \geq C(\mathbb{E}|X|^2)^{1/2}.$$

Corollary

For any log-concave vector X in $\mathbb{R}^n$, any Euclidean norm $\| \ \|$ on $\mathbb{R}^n$ and $p \geq 1$ we have

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\Big((\mathbb{E}\|X\|^2)^{1/2} + \sup_{\|t\|_* \leq 1} (\mathbb{E}|\langle t, X\rangle|^p)^{1/p}\Big), \qquad (1)$$

where $(\mathbb{R}^n, \|\cdot\|_*)$ is a dual space to $(\mathbb{R}^n, \|\cdot\|)$.

It is an open problem whether (1) holds for arbitrary norms

Bounds with use of weak parameter

Theorem

For any n-dimensional log-concave isotropic vector X,

$$\mathbb{P}(X_l^* \geq t) \leq \exp\Big(-\sigma_X^{-1}\Big(\frac{1}{C}t\sqrt{l}\Big)\Big) \quad \textit{for } t \geq C\log\Big(\frac{en}{l}\Big).$$

As before the proof is based on a suitable estimate of N_X:

Theorem

Let X be an isotropic log-concave vector in $\mathbb{R}^n$. Then

$$\mathbb{E}(t^2 N_X(t))^p \leq (C\sigma_X(p))^{2p} \quad \textit{for } p \geq 2, t \geq C\log\Big(\frac{nt^2}{\sigma_X^2(p)}\Big).$$

Uniform bound for projections with weak parameter

Theorem

Let X be an isotropic log-concave vector in $\mathbb{R}^n$. Then for any $t \geq 1$,

$$\mathbb{P}\left(\sup_{|I|=m} |P_I X| \geq Ct\sqrt{m}\log\left(\frac{en}{m}\right)\right) \leq \exp\left(-\sigma_X^{-1}\left(\frac{t\sqrt{m}\log\left(\frac{en}{m}\right)}{\sqrt{\log(em/m_0)}}\right)\right),$$

where

$$m_0 = m_0(X,t) = \sup\left\{k \leq m\colon\ k\log\left(\frac{en}{k}\right) \leq \sigma_X^{-1}\left(t\sqrt{m}\log\left(\frac{en}{m}\right)\right)\right\}.$$

Weak parameter for convolution of log-concave measures

Proposition

Let $X^{(1)}, \ldots, X^{(d)}$ be independent isotropic log-concave vectors and $Y = \sum_{i=1}^{d} x_i X^{(i)}$. Then

$$\sigma_Y(p) \leq C(\sqrt{p}|x| + p\|x\|_\infty). \quad \text{for } p \geq 2.$$

Sketch of the proof. Fix $t \in S^{n-1}$. Let E_i be independent symmetric exponential random variables with variance 1. The result of Borell gives $\mathbb{E}|\langle t, X^{(i)}\rangle|^p \leq C^p \mathbb{E}|E_i|^p$ for $p \geq 1$. Hence

$$(\mathbb{E}|\langle t, Y\rangle|^p)^{1/p} = \Big(\mathbb{E}\Big|\sum_{i=1}^{d} x_i \langle t, X^{(i)}\rangle\Big|^p\Big)^{1/p} \leq C\Big(\mathbb{E}\Big|\sum_{i=1}^{d} x_i E_i\Big|^p\Big)^{1/p}$$
$$\leq C(\sqrt{p}|x| + p\|x\|_\infty),$$

where the last inequality follows by the Gluskin and Kwapień bound. □

Order statistics of convolutions

Corollary

Let $X^{(1)}, \ldots, X^{(m)}$ *be independent isotropic log-concave vectors and* $Y = \sum_{i=1}^{m} x_i X^{(i)}$. *Then*

$$\mathbb{P}(Y_l^* \geq t) \leq \exp\Big(-\frac{1}{C}\min\Big\{\frac{t^2 l}{|x|^2}, \frac{t\sqrt{l}}{\|x\|_\infty}\Big\}\Big) \quad \textit{for } t \geq |x| \log\Big(\frac{en}{l}\Big).$$

Uniform bound for projections of convolutions

Theorem

Let $Y = \sum_{i=1}^{d} x_i X^{(i)}$, where $X^{(1)}, \ldots, X^{(d)}$ are independent isotropic n-dimensional log-concave vectors. Assume that $|x| \leq 1$ and $\|x\|_\infty \leq b \leq 1$.

i) If $b \geq \frac{1}{\sqrt{m}}$, then for any $t \geq 1$,

$$\mathbb{P}\left(\sup_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=m}} |P_I Y| \geq Ct\sqrt{m} \log\left(\frac{en}{m}\right) \right) \leq \exp\left(-\frac{t\sqrt{m}\log\left(\frac{en}{m}\right)}{b\sqrt{\log(e^2 b^2 m)}} \right).$$

ii) If $b \leq \frac{1}{\sqrt{m}}$ then for any $t \geq 1$,

$$\mathbb{P}\left(\sup_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=m}} |P_I Y| \geq Ct\sqrt{m} \log\left(\frac{en}{m}\right) \right)$$

$$\leq \exp\left(-\min\left\{ t^2 m \log^2\left(\frac{en}{m}\right), \frac{t}{b}\sqrt{m} \log\left(\frac{en}{m}\right) \right\} \right).$$

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