Quantum Optimal Control Landscapes

— a "Simplicity" Theory —

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BIRS Workshop on Quantum Control
Banff, April 5, 2011

Outline

Schemes for ultrafast laser control

- Frequency-domain approach: Two-pathway interference
- Time-domain approach: Pump-dump, STIRAP
- Optimal design approach: Optimal control theory, leaning control

C. Brif, R. Chakrabarti and H. Rabitz, "Control of quantum phenomena: past, present and future", New J. Phys. 12 075008, 2010.

Achievements

Optimization is supposed to be hard due to

- Limited bandwidth and severe noise in shaped pulses;
- A large number of control parameters.

What have been reported:

- $>$ 1000 excellent simulation results (since 1985);
- $\sim$ 150 successful close-loop experiments (since 1998).

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Why is it easy to find a good quantum control?

Quantum Control Landscape: basic concepts

Definition: the graph of the mapping from the control variables to the cost functional.

H. Rabitz, M. Hsieh and C. Rosenthal, Science, 303: 1998-2001, 2004;
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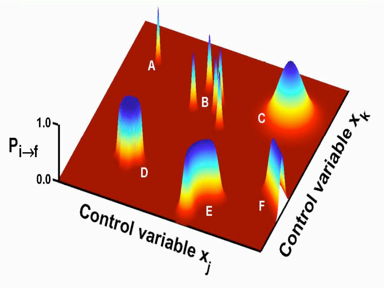

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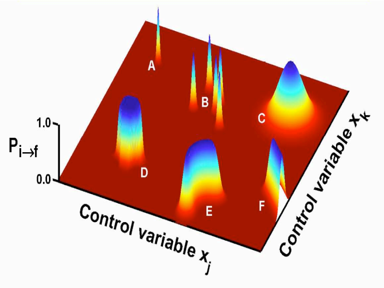

Critical topology: the topology of the set of critical points.

- Distribution of candidate solutions — algorithmic efficiency.
- Multiplicity of optimal solution set — robustness.

What we like...

What we dislike...

Control landscape for Observable Preparation

Schrödinger equation for an N-level closed quantum system:

$$\frac{\partial}{\partial t}\rho(t) = \frac{1}{i\hbar}\Big[H_0 - \epsilon(t)\mu, \rho(t)\Big], \quad \rho(t_0) = \rho_0.$$

where $\epsilon(\cdot)$ is the control field. Consider the maximization of $\langle O \rangle$ at $t = T$:

$$J[\epsilon(\cdot)] = \mathrm{Tr}\{\rho[T;\epsilon(\cdot)]O\}, \quad \epsilon(\cdot) \textit{ admissible},$$

In principle, what does the landscape look like under unlimited control resources?

Control landscape at a coarse-grained scale

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Projection from the **dynamical control landscape**

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onto the **kinematic control landscapes**:

$$\begin{aligned} J(\rho) &= \mathrm{Tr}(\rho O), & \rho \textit{ achievable}. \\ J(U) &= \mathrm{Tr}(U\rho_0 U^\dagger O), & U \textit{ achievable}. \end{aligned}$$

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where U is the propagator at $t = T$. In the case that the system is **controllable**

$$J(U) = \mathrm{Tr}(U\rho_0 U^\dagger O), \quad U \in \mathcal{U}(N).$$

Question

Dynamical control landscape
high-dimensional and highly nonlinear.

Kinematic control landscape
lower-dimensional and linear/quadratic.

What can be learned about the dynamical landscape from the kinematic one?

Landscape Reduction

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Conclusion: critical topology preserved from the dynamical to the kinematic picture if all admissible controls are regular.

Conditions for kinematic landscape critical points

Take the parametrization $U \to Ue^{isA}$ in $\mathcal{U}(N)$ for any $A^\dagger = A$ and take the derivative of J:

$$\frac{\mathrm{d}J}{\mathrm{d}s}\Big|_{s=0} = \mathrm{Tr}(iA[U\rho_0 U^\dagger, O]) = 0, \quad \forall A^\dagger = A.$$

Critical Condition: $[U\rho_0 U^\dagger, O] = 0.$

In particular, when ρ and O are nondegenerate, the critical U simultaneously diagonalizes $\rho(T)$ and O. [a]

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- Robustness of optimal controls on the "flat top" (maximum submanifold).

Control landscape for unitary gate fidelity

Fidelity defined as the distance from a desired quantum gate:

$$J(U) = \|U - W\|^2 = 2N - 2Re\mathrm{Tr}(W^\dagger U), \quad U \in \mathcal{U}(N).$$

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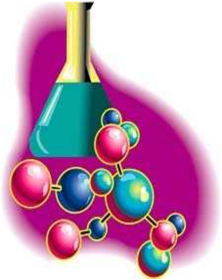
How about open quantum systems?

In reality, environmental interactions are always present:



$$H = H_S \otimes \mathbb{I}_\lambda + \mathbb{I}_N \otimes H_E + H_{SE}$$

Kinematic Control Landscape for Open Quantum Systems

Definition

$$J(\{K_j\}) = \sum_j \mathrm{Tr}(K_j \rho_0 K_j^\dagger O), \quad \sum_{j=1}^{\lambda} K_j^\dagger K_j = \mathbb{I}_N.$$

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Assumptions

- all Kraus maps are achievable;
- all admissible controls are regular.

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The equity $\sum_j K_j^\dagger K_j = \mathbb{I}_N$ implies that the following K is the first N columns of some enlarged unitary matrix:

$$K = \begin{pmatrix} K_1 \\ \vdots \\ K_\lambda \end{pmatrix} = U \begin{pmatrix} I_N \\ \vdots \\ 0_N \end{pmatrix}, \quad U = \begin{pmatrix} K_1 & \cdots & * \\ \vdots & \vdots & * \\ K_\lambda & \cdots & * \end{pmatrix}$$

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Auxiliary control landscape for "system" + "environment".

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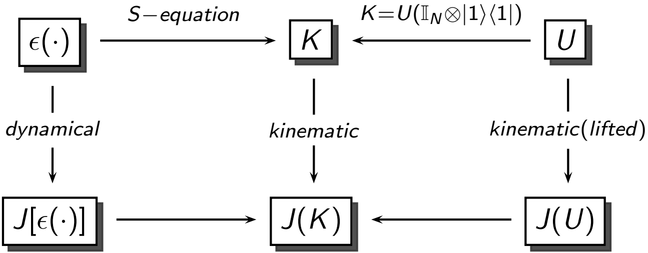

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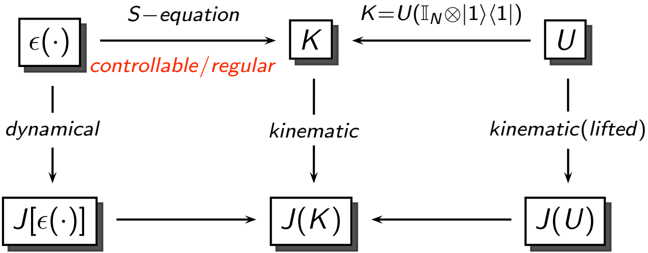

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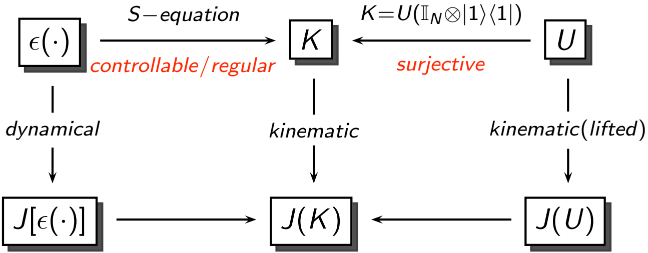

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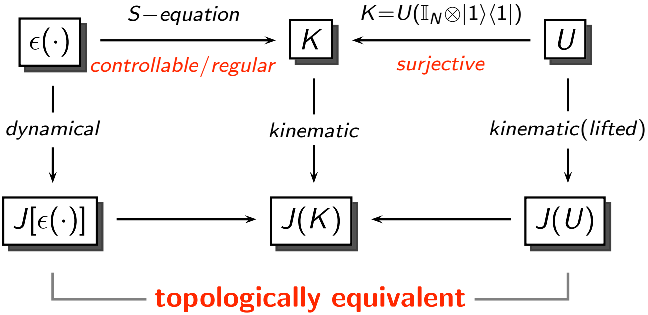

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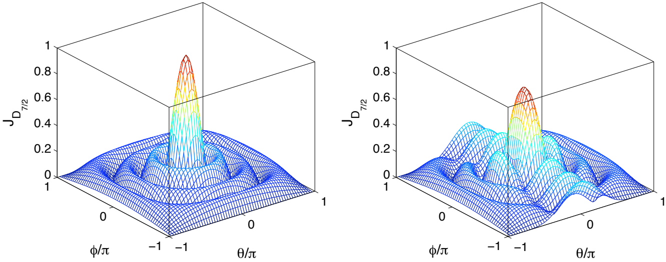
R. Wu, A. Pechen et al., J. Math. Phys., 49, 022108, 2008.

A. Pechen, D. Prokhorenko, et al., J. Phys. A: Math. Theor. 41, 045205 (2008)

Open question: the role of controllability?

Almost all quantum systems are controllable (C. Altafini, J. Math. Phys. 43, 2051 (2002).) BUT...



Gate fidelity landscape $J = \|\mathrm{Tr}(W^\dagger U)\|^2,\ U \in \mathcal{SU}(2) \subset \mathcal{U}(8)$.

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The role of Controllability beyond Yes-or-No

not only the existence of "wanted" controls but also nonexistence of "unwanted" controls

R. Wu, M. Hsieh and H. Rabitz, "The role of controllability in optimizing quantum dynamics", arXiv:0910.4702, 2010.

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Invisible critical points in the kinematic picture !

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Important in time optimal control (Lapert et al, PRL 2010) !

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Search efforts scaling with the system dimension and objectives, e.g., N, ρ, O or W?

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- A strong support for evident laboratory successes;
- Open up perspectives in developing more efficient algorithms (e.g., gradient and evolutionary-strategy algorithms are going on in Princeton laboratory).

THANK YOU !