Finite controllability of ∞-d quantum systems: Approximate controllability of trapped-ion quantum states

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thinking forward

Outline

- Work done in collaboration with Tony Bloch (Michigan) and Roger Brockett (Harvard)
- Finite controllability
- E.g.: spin-half coupled to QHO
  - Can show approx. controllability
- Two similar systems that are not finite controllable

Finite controllability

Definition

Given

-a system, and

-a nested set of finite dimensional subspaces

it will be said to be <u>finitely controllable</u> if

- it can be transferred from any point in one of the subspaces to any other point in that subspace

- with a trajectory lying entirely within the subspace.

Finite controllability theorem

Consider a complex Hilbert space X together with a nested set of finite-dimensional subsets

$$\mathrm{H} = \{\mathrm{H}_1 \subset \mathrm{H}_2 \subset \mathrm{H}_3 \cdots\}$$

Consider $$i\dot{\Psi} = \left(\sum_{i=1}^{m} u_i B_i\right)\Psi$$

where the B_i are Hermitian control operators.

Assume

- **H_1 is an invariant subspace for B_1**
- **the system is unit vector controllable on H_1 using only B_1**

Finite controllability theorem (cont'd)

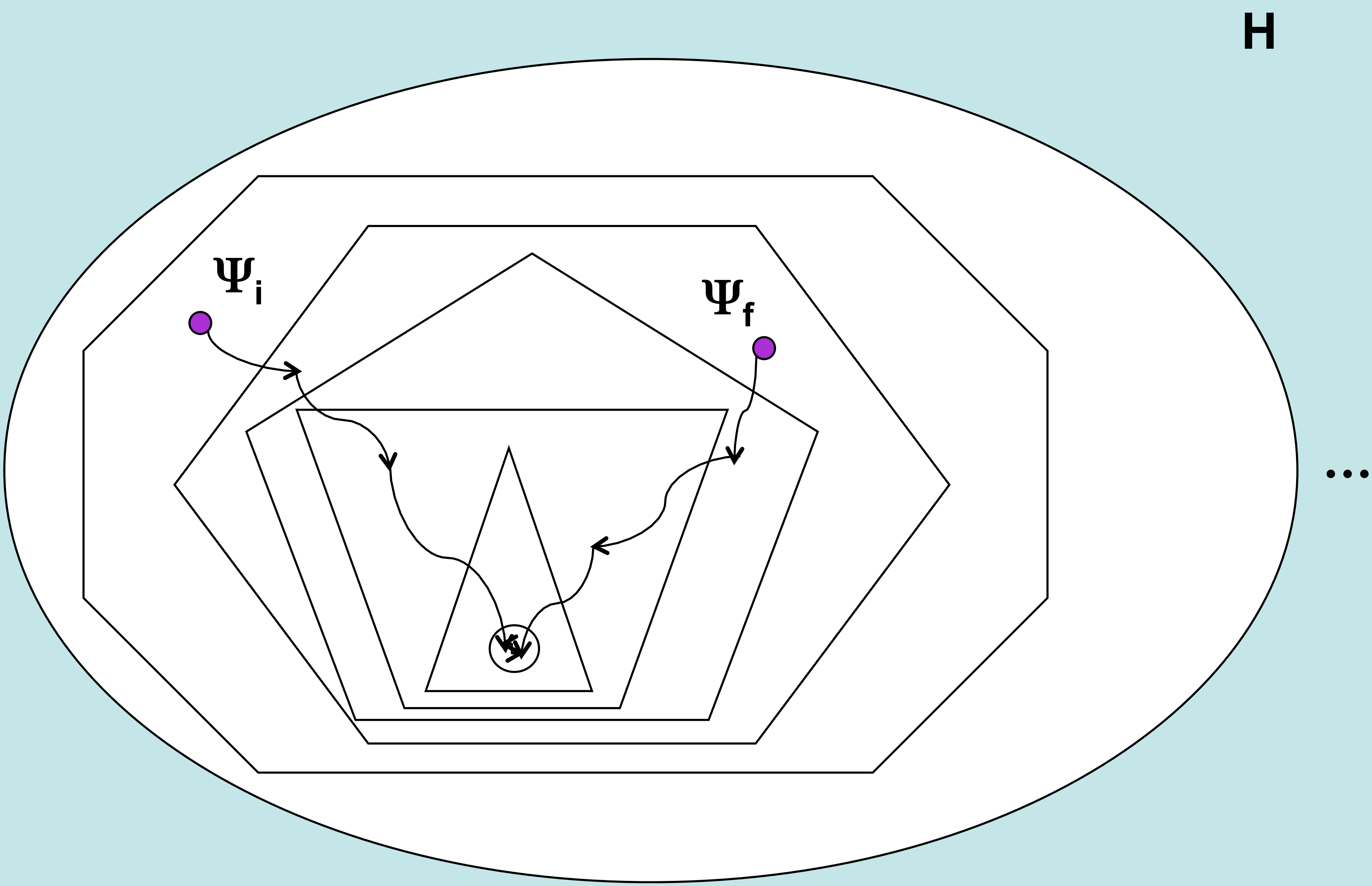
If

- **for each** $H_\alpha ; \alpha \neq 1$ **there is a** B_α **that leaves** H_α **invariant, and**
- **for any unit vector in** H_α **the orbit generated by** $\exp(iB_\alpha)$ **contains a point in one of the lower dimensional subspaces** H_β

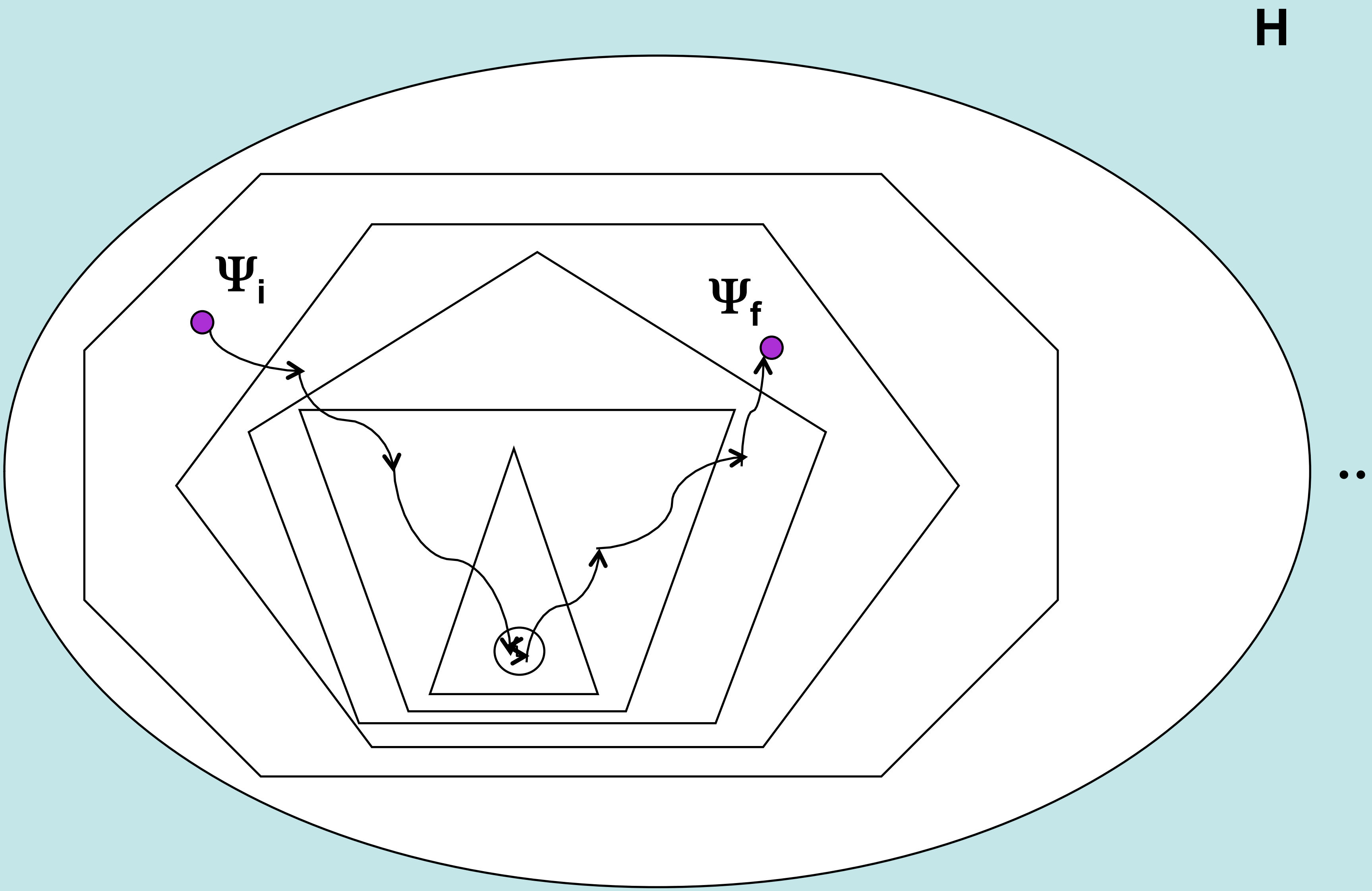
then any unit vector in any of the H_i **can be steered to any other unit vector in any other** H_j **using a finite number of piecewise constant controls.**

Bloch, Brockett, Rangan, quant-ph/0608075; ITAC 55, 1797 (2010)

Finite controllability

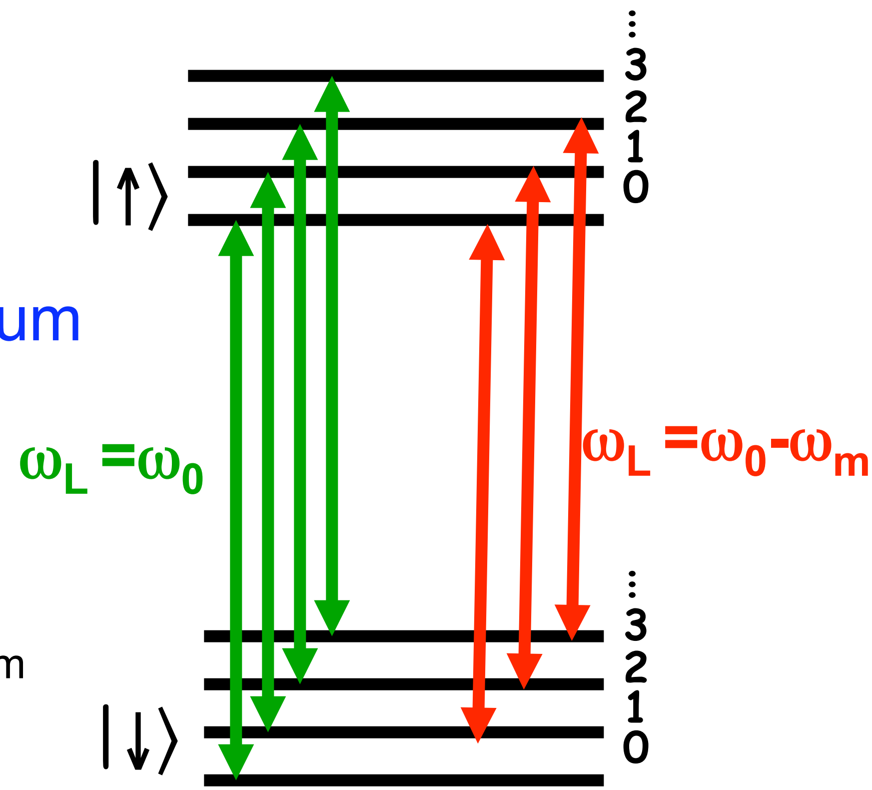


Explicit scheme



E.g.: trapped-ion quantum states

Physical system:
Trapped-ion
Model:
Spin ½ system coupled to quantum harmonic oscillator



Field-free Hamiltonian:
$H_0 = (^1/_2)\omega_0\sigma_z + \omega_m a^t a \quad ; \omega_0 >> \omega_m$

Control field:
$E(\xi,t) = \hat{\imath}\, E(t)\cos(k\xi - \omega_L t)$

Field-free eigenstates transitively connected by two resonant, monochromatic fields

Mathematical formulation

Interaction Hamiltonian:

$$H_I = (^1/_2)\,\mu\sigma_x\,E(t)\cos(k\xi_0(a+a^t)-\omega_L t)$$

$$= \sigma_x\,\Omega(t)\cos(\eta(a+a^t)-\omega_L t)$$

Lamb-Dicke parameter $\eta = k\xi_0$

Go into rotating frame, make RWA

Non-zero matrix elements in control Hamiltonian:
$\langle S,n|\,H_I\,|S',m\rangle$

Carrier ($\omega_L = \omega_0$):

$|\downarrow n\rangle \leftrightarrow |\uparrow n\rangle \sim L_n(\eta^2)$

First red sideband ($\omega_L = \omega_0 - \omega_m$):

$|\downarrow n\rangle \leftrightarrow |\uparrow n\text{-}1\rangle \sim i\,L^{(1)}{}_n(\eta^2)$

D. J. Wineland *et al., J. Res. Natl. Inst. Stand. Technol.* 103, 259 (1998).
D. Leibfried, R. Blatt, C. Monroe, D. Wineland. *Rev. Mod. Phys.* 75, 281 (2003).

Infinite Lie algebra

$$i\dot{\Psi} = (uB_c + vB_r)\Psi = (H_c + H_r)\Psi$$

$$\begin{pmatrix} 0 & A & 0 & 0 & 0 & \dots \\ A & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & A' & 0 & \dots \\ 0 & 0 & A' & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \dots \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & B & 0 & 0 & \dots \\ 0 & B & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & B' & \dots \\ 0 & 0 & 0 & B' & 0 & \dots \\ \dots \end{pmatrix}$$

$\exp(-iH\Delta t)$

$= \exp(-i(H_c+H_r)\Delta t)$

$= \exp(-iH_c\Delta t) \,.\, \exp(-iH_r\Delta t)$

$\quad .\exp(-1/2[H_c,H_r](\Delta t)^2)$

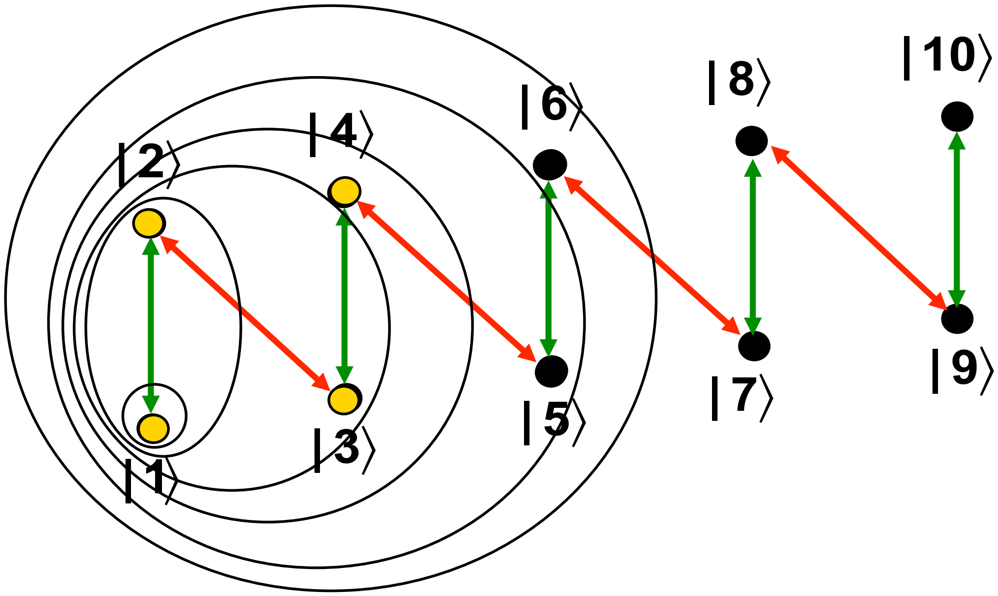
$\quad .\exp(1/12[H_c,[H_c,H_r]](\Delta t)^3)$

$\quad .\exp(1/12[[H_c,H_r],H_r](\Delta t)^3)\dots$

Lie algebra is ∞-D

The *alternate* application of control fields removes a chirp instability in unitary flows. (Brockett, Rangan, & Bloch, CDC 2003)

Finite controllability of trapped-ion



$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & B & 0 & 0 & \dots \\ 0 & B & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & B' & \dots \\ 0 & 0 & 0 & B' & 0 & \dots \\ \dots & & & & & \end{pmatrix}$$

Reachable set includes superpositions of finite numbers of eigenstates.
Approximate controllability of trapped-ion quantum states (BBR, quant-ph/0608075, ITAC 2010)

Also see: Ervedoza-Puel, Ann. I. H. Poincaré 26, 2111 (2009)

Lamb-Dicke limit

Transition matrix elements: $<\Phi_1| H_I |\Phi_2>$

Carrier:

$|\downarrow n\rangle \leftrightarrow |\uparrow n\rangle \sim L_n(\eta^2)$

First red sideband:

$|\downarrow n\rangle \leftrightarrow |\uparrow n\text{-}1\rangle \sim i\, L^{(1)}{}_n(\eta^2)$

In practical implementations, finding the control pulses is difficult (annoying) since the matrix elements depend on Laguerre polynomials

Lamb-Dicke limit (LDL): (motional cooling)

$\xi_0 \ll \lambda,\ \eta \ll 1$

Carrier:

$|\downarrow n\rangle \leftrightarrow |\uparrow n\rangle \sim 1$

First red sideband:

$|\downarrow n\rangle \leftrightarrow |\uparrow n\text{-}1\rangle \sim \sqrt{n}$

Explicit control schemes

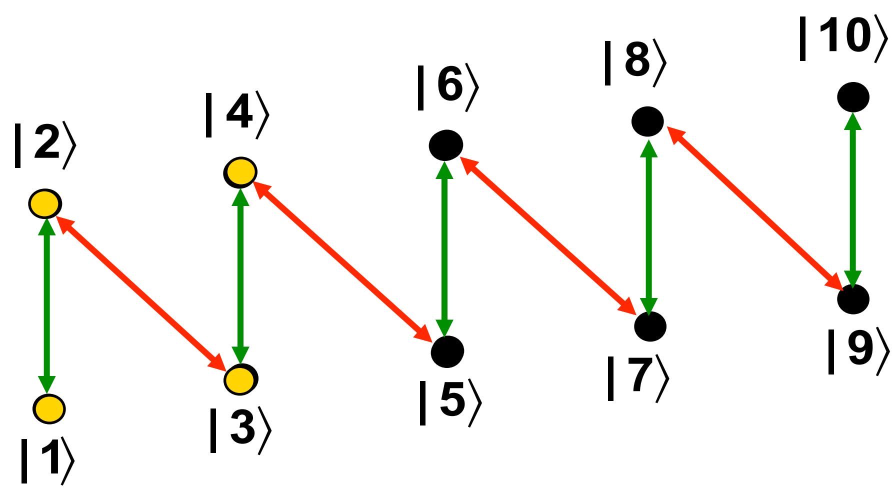
<u>In LDL</u>: **Law-Eberly scheme, PRL <u>76</u>, 1055 (1996)**

Kneer-Law scheme, PRA <u>57</u>, 2096 (1998)

<u>Beyond LDL</u>: Wei, Liu & Nori, PRA 70, 63801 (2004)

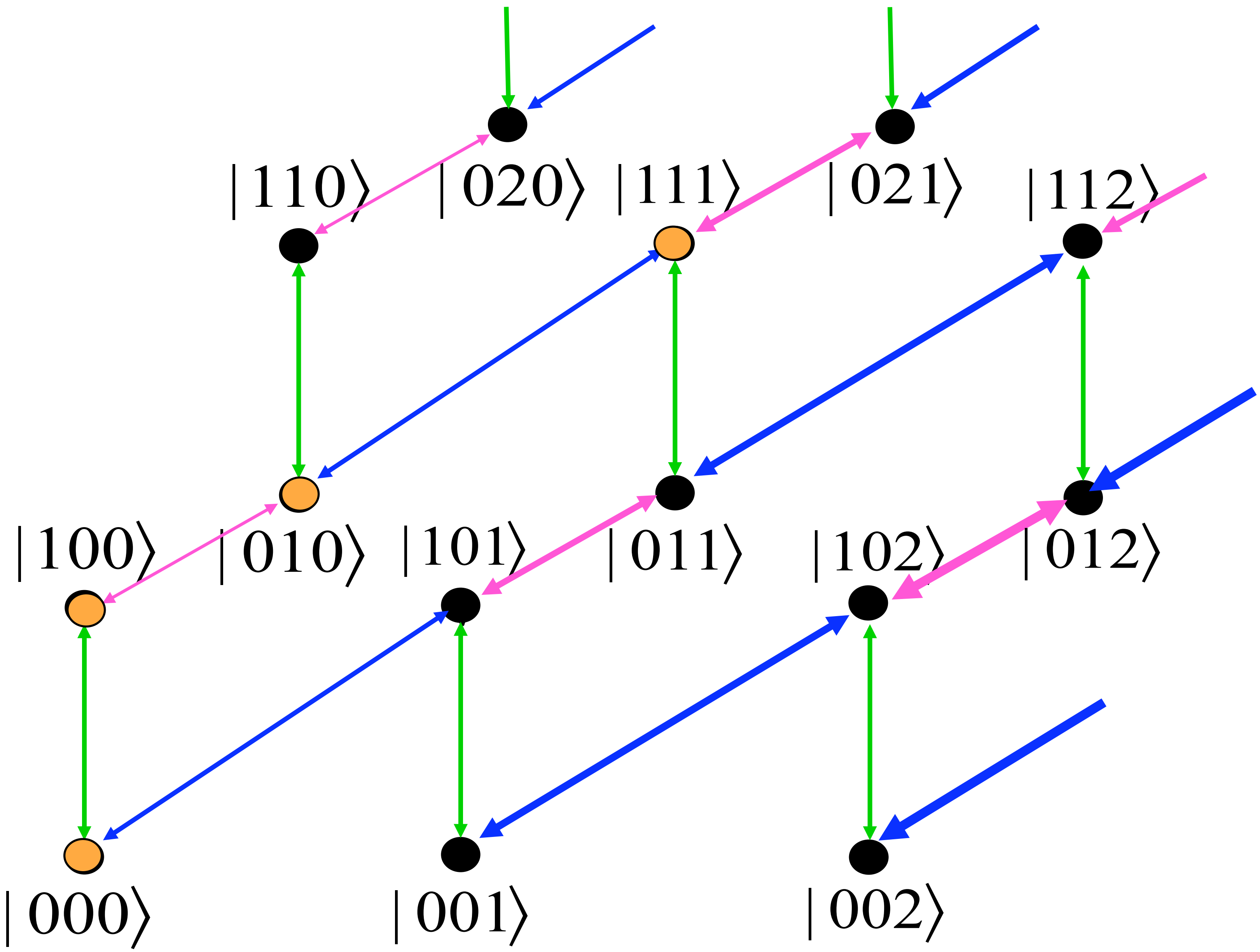
<u>Aim</u>: Start from ground state and create a finite superposition of trapped-ion energy eigenstates

<u>Method</u>: reverse engineer

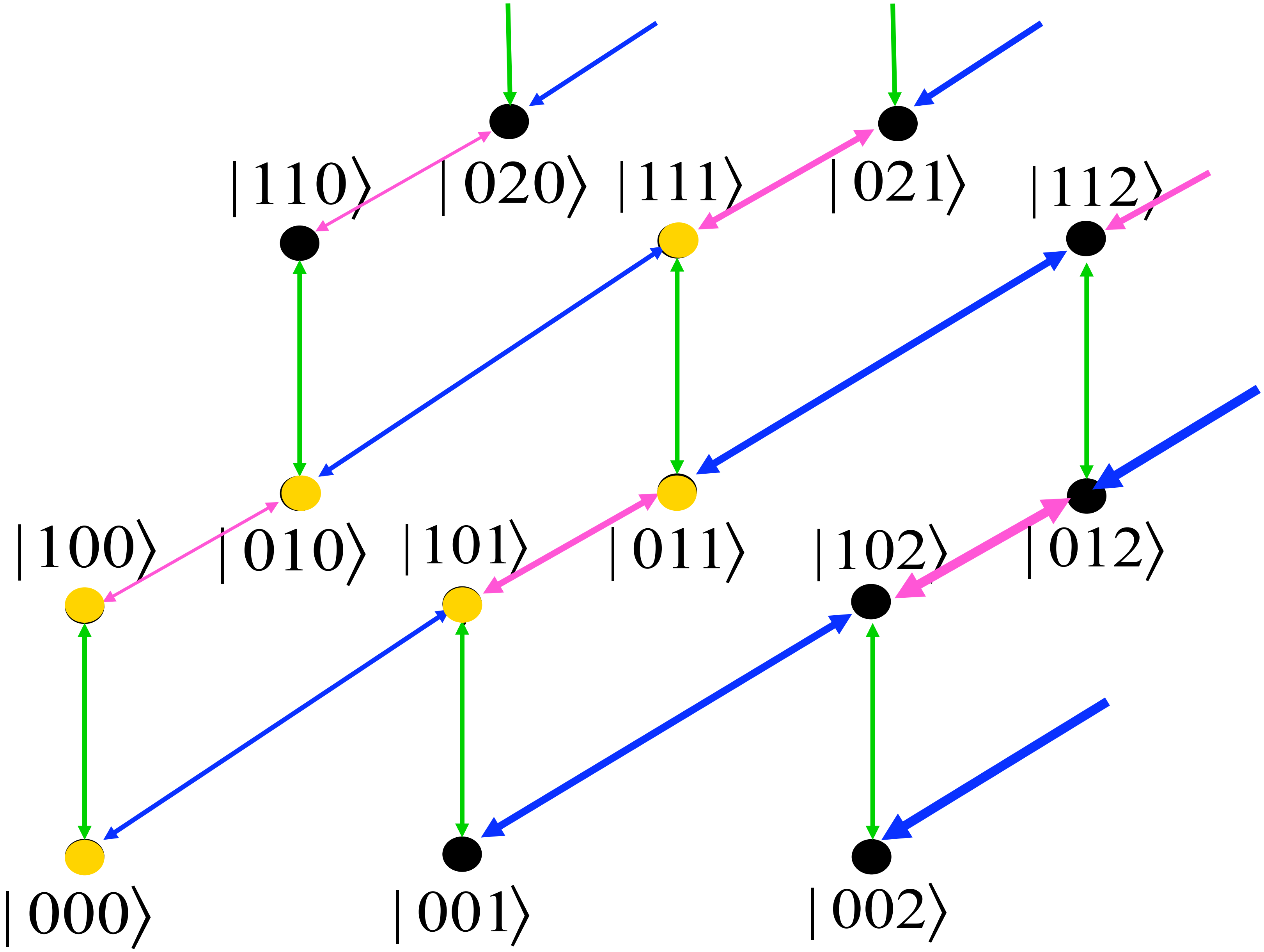


Experimental demonstration:
(LDL) Ben-Kish et al., Phys. Rev. Lett. <u>90</u>, 037902 (2003).
Without LDA (demonstration of principle):
B.E. King et al., Phys. Rev. Lett. <u>81</u>, 3631 (1998).

E.g.: spin-1/2 with 2 QHO's



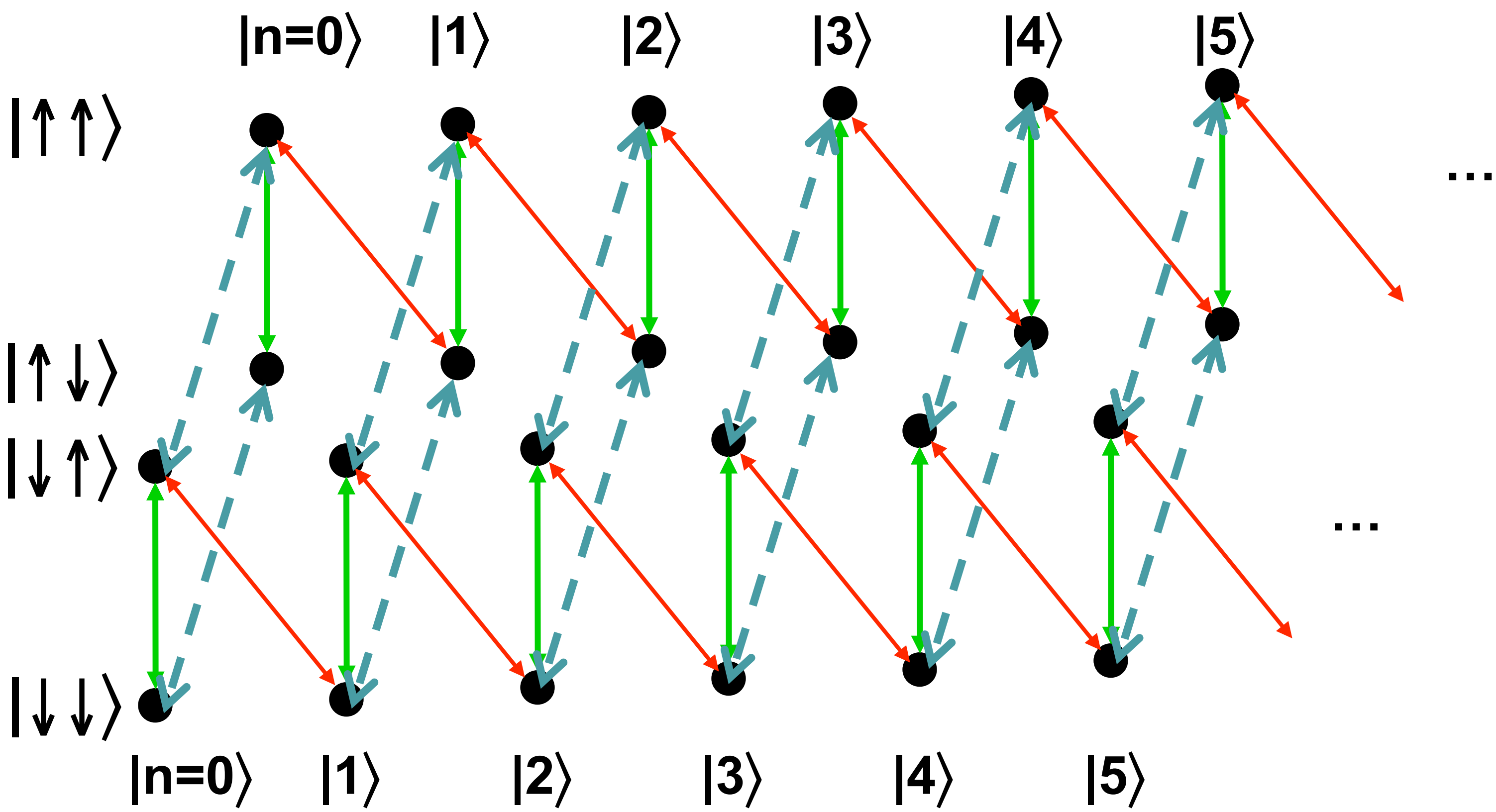
BUT - No finite controllability



E.g.: Two trapped-ions

Two spin ½ systems coupled to a quantum harmonic oscillator

Assume we have independent controls $u^{(1)}$, $u^{(2)}$ and $v^{(1)}$

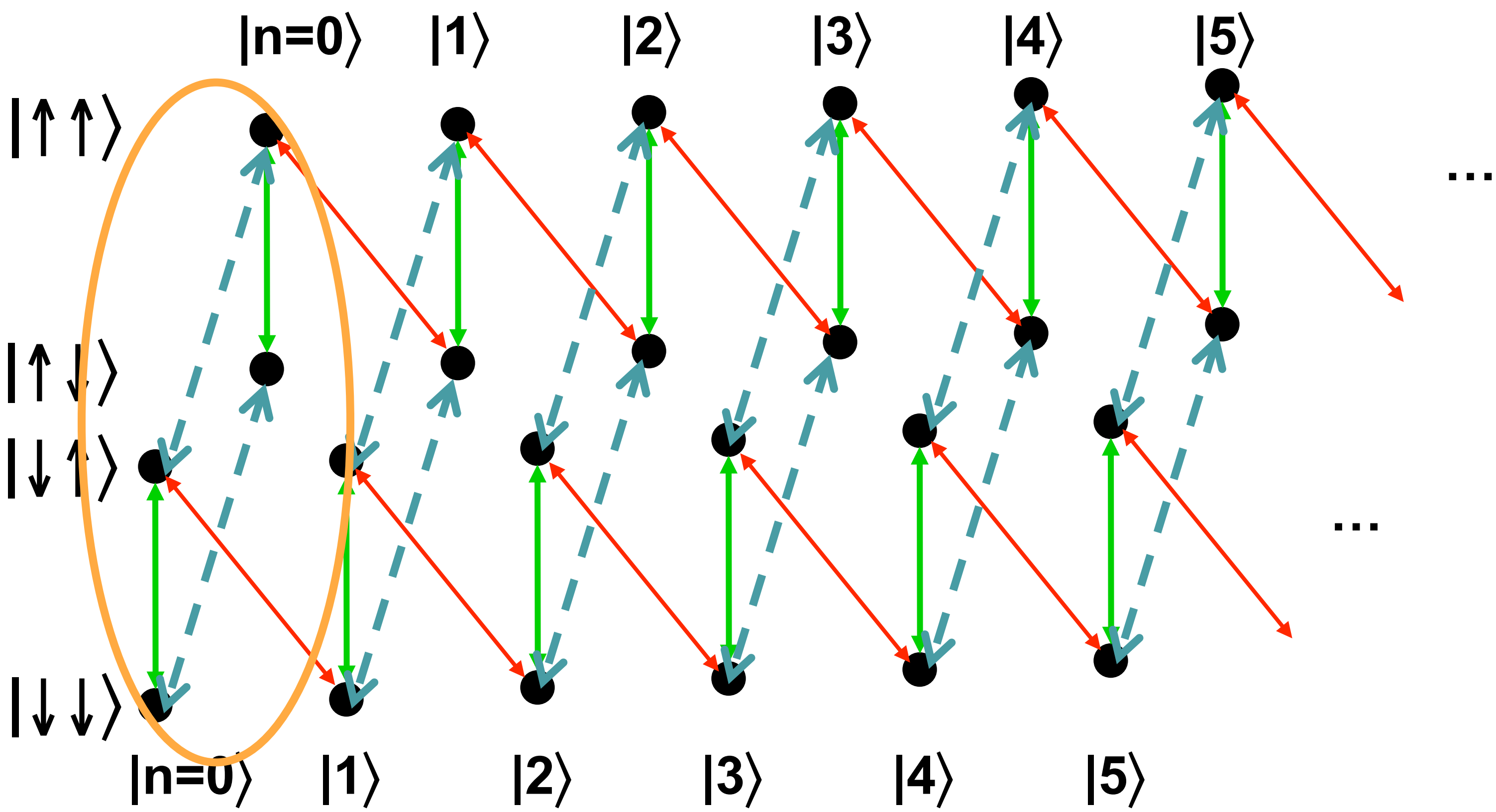


Have eigenstate controllability, can also make specific types of states: e.g.: Turquette et al., Phys. Rev. Lett. 81, 3631 (1998).

E.g.: Two trapped-ions

Two spin ½ systems coupled to a quantum harmonic oscillator

Assume we have independent controls $u^{(1)}$, $u^{(2)}$ and $v^{(1)}$

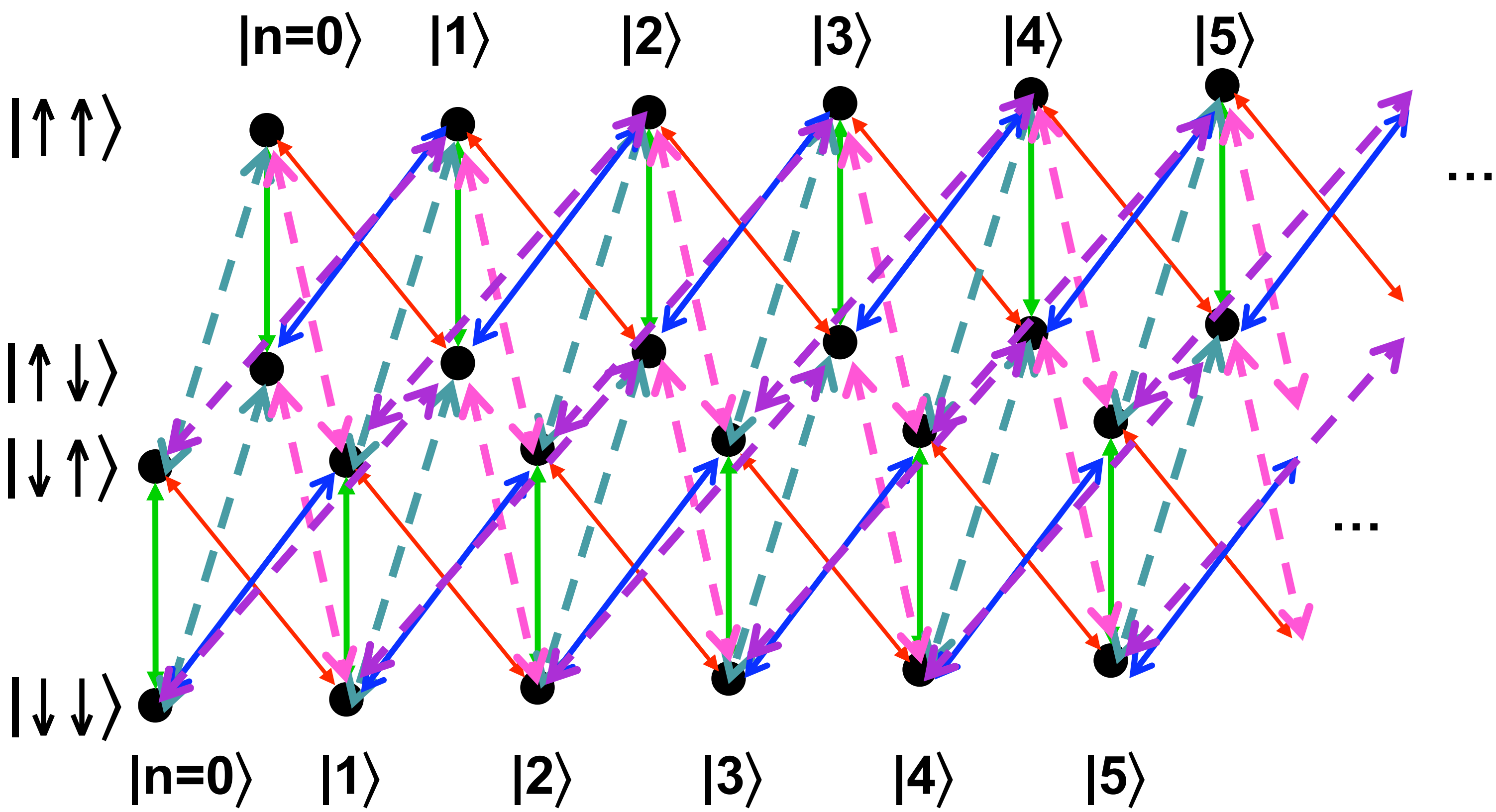


…but does not satisfy conditions for finite controllability.

Approximate controllability?

Two spin ½ systems coupled to a quantum harmonic oscillator

Assume we have 6 independent controls, still not fin. cont.



How to start from |↓↑0⟩/2 + √3 |↑↑0⟩/2 + √3|↑↓0⟩/2 and get |↓↓0⟩ using a single non-zero control at a time?

Summary

Finite controllability

-Approx. controllability of spin-half particle coupled to an harmonic oscillator

No finite controllability for

• Spin-half particle coupled to two harmonic oscillators

•Two spin-half particles coupled to an harmonic oscillator

- •Conjecture: the latter system is not approximately controllable using sequential fields

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