Controller design for infinite-dimensional systems

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Finite-dimensional and infinite-dimensional quantum systems

- for many quantum systems, the state-space is finite-dimensional
- e.g. controlling n-spins such as in NMR-based quantum computing
- in other systems, the state-space is infinite-dimensional
- for instance, where there is interaction between energy levels and/or spatial variation
- standard approach is to use several lowest energy levels
- avoid transition to higher levels

Subspace Invariance

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2},$$
$$\psi(0) = 0,\ \psi(1) = 0.$$

- $D(A) = \{\psi \in H^2(0,1), \psi(0) = \psi(1) = 0.\}$.
- A generates semigroup $S(t)$ on $L_2(0,1)$.

Consider $V = \{\psi \in L_2(0,1);\ \psi(r) = 0,\ 1/2 \leq x \leq 1]\}$.

$A : \big(D(A) \cap V\big) \Rightarrow V$
BUT
$\psi_0 \in V$ does NOT imply $S(t)\psi_0 \in V$

- spin-half particle coupled to 2 harmonic oscillators has the eigenstates invariant but not the trajectory (with respect to piecewise constant controls) (Bloch, Brockett and Rangan, 2010)

Generator and Semigroup Invariance

Generator Invariance

A subspace $V \subset H$ is A-invariant if $A(D(A) \cap V) \subset V$.

$$\Uparrow$$

Semigroup Invariance

A subspace V of H is semigroup invariant if $S(t)\psi_0 \subset V$ for all $\psi_0 \in V$.

Equivalent for finite-dimensional systems but not for infinite-dimensional systems.

Disturbance Decoupling

$$\frac{d\psi}{dt} = A\psi(t) + B\underbrace{u(t)}_{\textit{control}} + D\underbrace{v(t)}_{\textit{disturbance}},$$
$$y(t) = Cx(t)$$

Calculate feedback $u = K\psi$ so that $y(t) \equiv 0$.

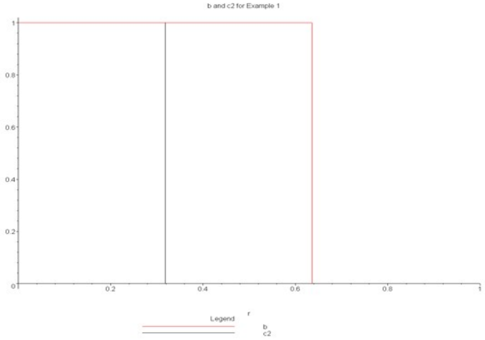
Problem is solvable $\Leftrightarrow$ there is a feedback K so that $A + BK$ generates a semigroup that is invariant on a closed subspace $V \subset \ker C$ where $d \subset V$.

Example of Disturbance Decoupling

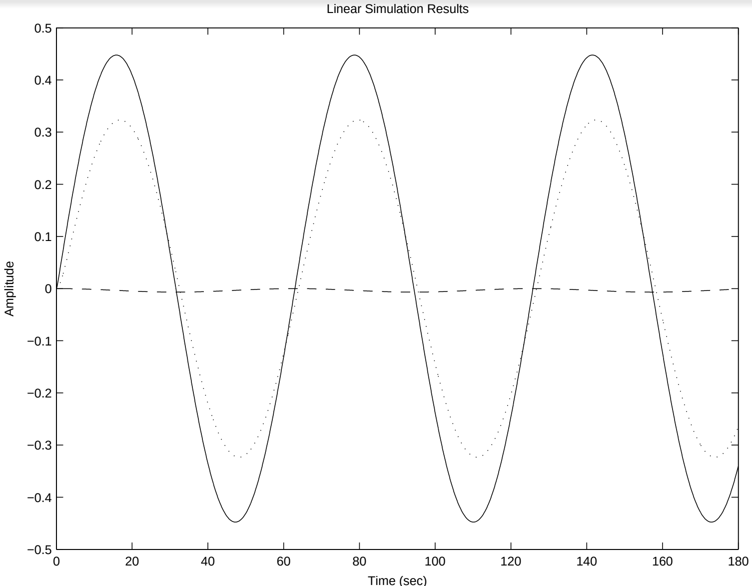
$$\frac{\partial \psi}{\partial t}(x,t) = \frac{\partial \psi^2}{\partial x^2}(x,t) + \chi_{[0,\frac{2}{\pi}]}(x)u(t) + \chi_{[1/2,1]}(x)v(t)$$

$$\psi(0,t) = 0, \qquad \psi(1,t) = 0$$

$$y(t) = \int_0^{\frac{1}{\pi}} \psi(x,t)dr.$$



Example of Disturbance Decoupling. Full (-) and reduced-order (...) control



Approximation of Schrödinger equation

$$\imath \frac{\partial \psi(x,t)}{\partial t} = \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t)u(t) \tag{1}$$
$$\psi(0,t) = \psi(1,t) = 0.$$

Let $\phi_n(x)$ be the orthonormal eigenfunctions of $\frac{\partial^2}{\partial x^2}$ and λ_n the associated eigenvalues.

If $u(t) \equiv 0$, approximate ψ by: $\tilde{\psi}(x,t) = \sum_{k=1}^{N} c_k e^{\imath \lambda_k t} \phi_k(x)$.

More generally:

$$\tilde{\psi}(x,t) = \sum_{k=1}^{N} c_k(t)\phi_k(x). \tag{2}$$

Finite-dimensional approximation - heat equation

Suppose instead of (1) we have

$$\frac{\partial \psi(x,t)}{\partial t} = \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)u(t). \qquad (3)$$

Substituting the approximation (2) into (3) and projecting the error onto the span of each ϕ_n, we obtain the system of o.d.e's

$$\dot{c}_n(t) = \lambda_n c_n(t) + \langle V, \phi_n \rangle u(t), \qquad n = 1..N$$

Each mode is decoupled, so neglecting $k > N$ does not affect the lower modes.

Finite-dimensional approximation - Schrödinger equation

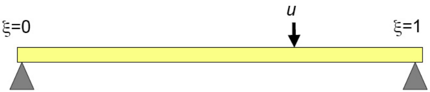
$$\imath \frac{\partial \psi(x,t)}{\partial t} = \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t)u(t)$$

Substituting the approximation (2) into (3) and projecting the error onto the span of each ϕ_n, we obtain the system of o.d.e's

$$\imath \dot{c}_n(t) = \lambda_n c_n(t) + u(t) \sum_{k=1}^{N} c_k(t) \langle V\phi_k, \phi_n \rangle .$$

- Even if the system is prepared so that the higher energy levels are zero, in general they will be activated and furthermore, will affect the lower modes.
- loss of probability

Example: Reduced-order Feedback Controller Design



PDE

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = b(x)u(t), \qquad t \geq 0,\ 0 < x < 1,$$

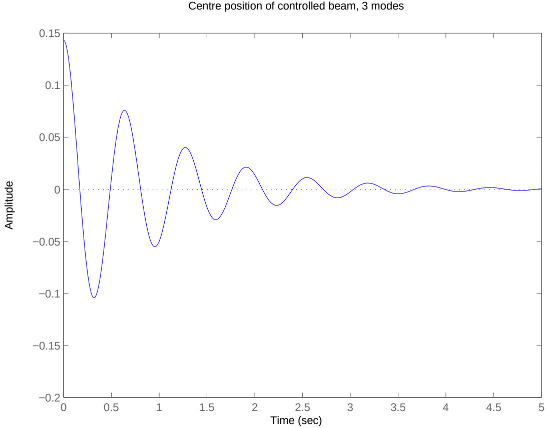
$$b(x) = \begin{cases} 1/\delta, & |x - .5| < \frac{\delta}{2} \\ 0, & |x - .5| \geq \frac{\delta}{2} \end{cases} .$$

$$w(0,t) = 0,\ w_{xx}(0,t) = 0,\ w(1,t) = 0, w_{xx}(1,t) = 0.$$

- Use eigenfunctions as basis for approximating subspace
- Linear quadratic regulator, state weight I, control weight 1
- Feedback controller is $u(t) = -B^*\Pi[w(t)\,\dot{w}(t)]$.

Design of Reduced-order Controller

- Use first 3 modes to design controller
- Initial condition is first eigenfunction.



Controlled system, 3 modes in approximated system

Design of Reduced-order Controller

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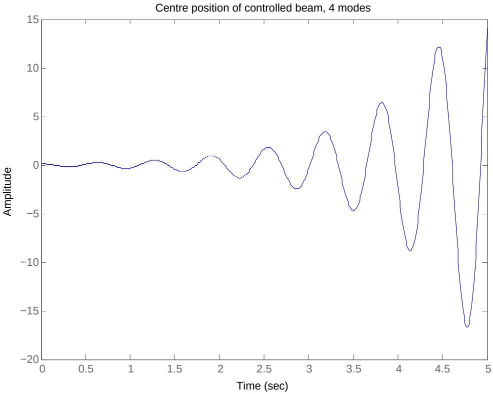

Controlled system, *4* modes in approximated system

Summary

- infinite-dimensional system behaviour can be fundamentally different from finite-dimensional
- in control, neglected modes can drastically affect the solution
- interaction stronger for bilinear systems than linear systems

Survey/tutorial papers on use of approximations in controller design

- H.T. Banks and R. H. Fabiano, “Approximation issues for applications in optimal control and parameter estimation”, *Modelling and computation for applications in mathematics, science, and engineering*, Numer. Math. Sci. Comput., Oxford Univ. Press, 141-165, 1998.
- K. A. Morris, “Control of Systems Governed by Partial Differential Equations”, ed. W. S. Levine, *The Control Theory Handbook* , CRC Press, 2010.
- E. Zuazua, “Propagation, observation, and control of waves approximated by finite difference methods”, SIAM Rev., 47-2:197-243,2005.

Recent papers on invariance for infinite-dimensional systems

- A. M. Bloch, R.W. Brockett, C. Rangan, "The Controllability of Infinite Quantum Systems and Closed Subspace Criteria", *IEEE Trans. Auto. Cont.*, 55-8: 1797 - 1805, 2010. (See also Chitra Rangan's talk.)
- K.A. Morris and R. Rebarber, "Feedback Invariance of SISO Infinite-Dimensional Systems", *Mathematics of Control, Signals and Systems*, Vol. 19, pg. 313-335, 2007. (Has fairly complete set of references.)
- K.A. Morris and R. E. Rebarber, "Zeros of SISO Infinite-Dimensional Systems", *International Journal of Control*, to appear.