Frenet-Serret and the Estimation of Curvature and Torsion

Peter Kim

University of Guelph

August 29, 2011
BIRS

Joint work with Kang-Rae Kim (Korea University), Ja-Yong Koo (Korea University) and Michael Pierrynowski (McMaster University)

Outline

Preliminaries

Outline

Preliminaries

Curvature and Torsion

Outline

Outline

Preliminaries

Curvature and Torsion

Statistical Estimation

Application to Biomechanics

Preliminaries

differentiable curve $\alpha : [a, b] \longrightarrow \mathbb{R}^3$

Preliminaries

differentiable curve $\alpha : [a, b] \to \mathbb{R}^3$

curve is parameterized by arclength $\|\alpha'(t)\| = 1$ for $t \in [a, b]$

Preliminaries

differentiable curve $\alpha : [a, b] \rightarrow \mathbb{R}^3$

curve is parameterized by arclength $\|\alpha'(t)\| = 1$ for $t \in [a, b]$

$T = \alpha'$ the tangent

Preliminaries

differentiable curve $\alpha : [a, b] \to \mathbb{R}^3$

curve is parameterized by arclength $\|\alpha'(t)\| = 1$ for $t \in [a, b]$

$T = \alpha'$ the tangent

$N = T'/\|T'\|$ the normal

Preliminaries

differentiable curve $\alpha : [a, b] \to \mathbb{R}^3$

curve is parameterized by arclength $\|\alpha'(t)\| = 1$ for $t \in [a, b]$

$T = \alpha'$ the tangent

$N = T'/\|T'\|$ the normal

$B = T \times N$ the binormal

Preliminaries

differentiable curve $\alpha : [a, b] \rightarrow \mathbb{R}^3$

curve is parameterized by arclength $\|\alpha'(t)\| = 1$ for $t \in [a, b]$

$T = \alpha'$ the tangent

$N = T'/\|T'\|$ the normal

$B = T \times N$ the binormal

where $'$ indicates differentiation and $\times$ denotes cross product

Preliminaries–Frenet-Serret

$\{T, N, B\}$ are pairwise orthonormal hence form an orthonormal basis for $\mathbb{R}^3$ for each $t \in [a, b]$

Preliminaries–Frenet-Serret

$\{T, N, B\}$ are pairwise orthonormal hence form an orthonormal basis for $\mathbb{R}^3$ for each $t \in [a, b]$

Frenet-Serret frame

Preliminaries–Frenet-Serret

$\{T, N, B\}$ are pairwise orthonormal hence form an orthonormal basis for $\mathbb{R}^3$ for each $t \in [a, b]$

Frenet-Serret frame satisfies the differential equation

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Preliminaries–Frenet-Serret

$\{T, N, B\}$ are pairwise orthonormal hence form an orthonormal basis for $\mathbb{R}^3$ for each $t \in [a, b]$

Frenet-Serret frame satisfies the differential equation

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Frenet-Serret formulas

Curvature and Torsion

$\kappa(t) = \|T'(t)\|$ is the curvature

Curvature and Torsion

$\kappa(t) = \|T'(t)\|$ is the curvature

$\tau(t) = -\langle N(t), B'(t)\rangle$ is the torsion

Curvature and Torsion

$\kappa(t) = \|T'(t)\|$ is the curvature

$\tau(t) = -\langle N(t), B'(t)\rangle$ is the torsion

curvature measures how fast the unit tangent vector rotates

Curvature and Torsion

$\kappa(t) = \|T'(t)\|$ is the curvature

$\tau(t) = -\langle N(t), B'(t)\rangle$ is the torsion

curvature measures how fast the unit tangent vector rotates

torsion characterizes the non-planarity of a three-dimensional space curve

Curvature and Torsion

$\kappa(t) = \|T'(t)\|$ is the curvature

$\tau(t) = -\langle N(t), B'(t)\rangle$ is the torsion

curvature measures how fast the unit tangent vector rotates

torsion characterizes the non-planarity of a three-dimensional space curve

Curvature and Torsion - non unit speed

previous formula assumes unit speed, if we take the original curve $\alpha : [a, b] \rightarrow \mathbb{R}^3$

Curvature and Torsion - non unit speed

previous formula assumes unit speed, if we take the original curve $\alpha : [a, b] \to \mathbb{R}^3$

$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3}$ is the curvature

Curvature and Torsion - non unit speed

previous formula assumes unit speed, if we take the original curve $\alpha : [a, b] \to \mathbb{R}^3$

$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3}$ is the curvature

$\tau(t) = -\frac{\langle \alpha'(t) \times \alpha''(t), \alpha'''(t) \rangle}{\|\alpha'(t) \times \alpha''(t)\|^2}$ is the torsion

Curvature and Torsion - non unit speed

previous formula assumes unit speed, if we take the original curve $\alpha : [a, b] \to \mathbb{R}^3$

$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3}$ is the curvature

$\tau(t) = -\frac{\langle \alpha'(t) \times \alpha''(t), \alpha'''(t) \rangle}{\|\alpha'(t) \times \alpha''(t)\|^2}$ is the torsion

curvature measures how fast the tangent vector rotates

Curvature and Torsion - non unit speed

previous formula assumes unit speed, if we take the original curve $\alpha : [a, b] \to \mathbb{R}^3$

$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3}$ is the curvature

$\tau(t) = -\frac{\langle \alpha'(t) \times \alpha''(t), \alpha'''(t) \rangle}{\|\alpha'(t) \times \alpha''(t)\|^2}$ is the torsion

curvature measures how fast the tangent vector rotates

torsion characterizes the non-planarity of a three-dimensional space curve

Curvature and Torsion - non unit speed

previous formula assumes unit speed, if we take the original curve $\alpha : [a, b] \to \mathbb{R}^3$

$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3}$ is the curvature

$\tau(t) = -\frac{\langle \alpha'(t) \times \alpha''(t), \alpha'''(t) \rangle}{\|\alpha'(t) \times \alpha''(t)\|^2}$ is the torsion

curvature measures how fast the tangent vector rotates

torsion characterizes the non-planarity of a three-dimensional space curve

Statistical Estimation

consider $h_i = f(t_i) + \varepsilon_i$ using data (t_i, h_i) and assuming ε_i are errors , $i = 1, \ldots, n$

Statistical Estimation

consider $h_i = f(t_i) + \varepsilon_i$ using data (t_i, h_i) and assuming ε_i are errors , $i = 1, \ldots, n$

consider K knots, $\xi_1 < \xi_2 < \ldots < \xi_K$, and take B-spline basis functions, $B_j, j = 1, \ldots, K$

Statistical Estimation

consider $h_i = f(t_i) + \varepsilon_i$ using data (t_i, h_i) and assuming ε_i are errors , $i = 1, \ldots, n$

consider K knots, $\xi_1 < \xi_2 < \ldots < \xi_K$, and take B-spline basis functions, $B_j, j = 1, \ldots, K$

then form $s(t;\beta) = \sum_{j=1}^{K} \beta_j B_j(t)$

Statistical Estimation

consider $h_i = f(t_i) + \varepsilon_i$ using data (t_i, h_i) and assuming ε_i are errors , $i = 1, \ldots, n$

consider K knots, $\xi_1 < \xi_2 < \ldots < \xi_K$, and take B-spline basis functions, $B_j, j = 1, \ldots, K$

then form $s(t;\beta) = \sum_{j=1}^{K} \beta_j B_j(t)$

estimate β by

$$\hat{\beta} = \arg\min_{\beta} \sum_{i=1}^{n} \Big(h_i - \beta^\top B(t_i)\Big)^2,$$

Statistical Estimation

consider $h_i = f(t_i) + \varepsilon_i$ using data (t_i, h_i) and assuming ε_i are errors , $i = 1, \ldots, n$

consider K knots, $\xi_1 < \xi_2 < \ldots < \xi_K$, and take B-spline basis functions, $B_j, j = 1, \ldots, K$

then form $s(t;\beta) = \sum_{j=1}^{K} \beta_j B_j(t)$

estimate β by

$$\hat{\beta} = \arg\min_{\beta} \sum_{i=1}^{n} \left(h_i - \beta^\top B(t_i)\right)^2,$$

the estimator of f is defined by $\hat{f} = \sum_{j=1}^{K} \hat{\beta}_j B_j$

Model Selection

since we do not know which basis functions are useful in fitting, we adopt a model selection procedure as follows:

Model Selection

since we do not know which basis functions are useful in fitting, we adopt a model selection procedure as follows:

set K_0 initial knots with equally spaced percentiles among the data points

Model Selection

since we do not know which basis functions are useful in fitting, we adopt a model selection procedure as follows:

set K_0 initial knots with equally spaced percentiles among the data points

sequentially delete the basis function having the minimum absolute t-statistic

Model Selection

since we do not know which basis functions are useful in fitting, we adopt a model selection procedure as follows:

set K_0 initial knots with equally spaced percentiles among the data points

sequentially delete the basis function having the minimum absolute t-statistic

select the model having the minimum Akaike information criterion (AIC) statistic

Estimation of curvature and torsion

let $\alpha_n(t) = (\alpha_{1,n}(t), \alpha_{2,n}(t), \alpha_{3,n}(t))^\top$ be the fitted curve

Estimation of curvature and torsion

let $\alpha_n(t) = (\alpha_{1,n}(t), \alpha_{2,n}(t), \alpha_{3,n}(t))^\top$ be the fitted curve

define $\alpha_n^{(\ell)}(t) = (\alpha_{1,n}^{(\ell)}(t), \alpha_{2,n}^{(\ell)}(t), \alpha_{3,n}^{(\ell)}(t))^\top$ be the $\ell-$th derivative

Estimation of curvature and torsion

let $\alpha_n(t) = (\alpha_{1,n}(t), \alpha_{2,n}(t), \alpha_{3,n}(t))^\top$ be the fitted curve

define $\alpha_n^{(\ell)}(t) = (\alpha_{1,n}^{(\ell)}(t), \alpha_{2,n}^{(\ell)}(t), \alpha_{3,n}^{(\ell)}(t))^\top$ be the $\ell-$th derivative

alternatively one may start with an $\ell-$th derivative and integrate to get lower order derivatives

Estimation of curvature and torsion

let $\alpha_n(t) = (\alpha_{1,n}(t), \alpha_{2,n}(t), \alpha_{3,n}(t))^\top$ be the fitted curve

define $\alpha_n^{(\ell)}(t) = (\alpha_{1,n}^{(\ell)}(t), \alpha_{2,n}^{(\ell)}(t), \alpha_{3,n}^{(\ell)}(t))^\top$ be the $\ell-$th derivative

alternatively one may start with an $\ell-$th derivative and integrate to get lower order derivatives

$\hat{\kappa}_n(t) = \frac{\|\alpha_n'(t) \times \alpha_n''(t)\|}{\|\alpha_n'(t)\|^3}$ is the curvature estimator

Estimation of curvature and torsion

let $\alpha_n(t) = (\alpha_{1,n}(t), \alpha_{2,n}(t), \alpha_{3,n}(t))^\top$ be the fitted curve

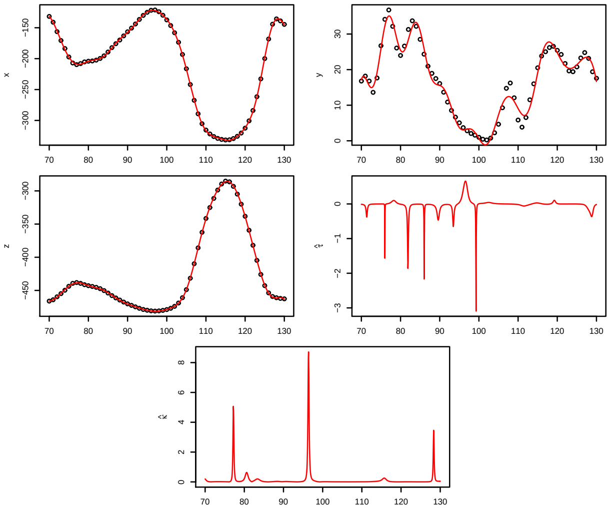
define $\alpha_n^{(\ell)}(t) = (\alpha_{1,n}^{(\ell)}(t), \alpha_{2,n}^{(\ell)}(t), \alpha_{3,n}^{(\ell)}(t))^\top$ be the $\ell-$th derivative

alternatively one may start with an $\ell-$th derivative and integrate to get lower order derivatives

$\hat{\kappa}_n(t) = \frac{\|\alpha_n'(t) \times \alpha_n''(t)\|}{\|\alpha_n'(t)\|^3}$ is the curvature estimator

$\hat{\tau}_n(t) = -\frac{\langle \alpha_n'(t) \times \alpha_n''(t), \alpha_n'''(t)\rangle}{\|\alpha_n'(t) \times \alpha_n''(t)\|^2}$ is the torsion estimator

Knee data


x
−150
−200
−250
−300
70
80
90
100
110
120
130
y
0
10
20
30
z
−300
−350
−400
−450
$\hat{\tau}$
0
−1
−2
−3
$\hat{\kappa}$
0
2
4
6
8