Understanding the (total) mass of Quadratic forms of fixed determinant

(Jonathan Hanke)

Understanding the (total) mass of quadratic forms of fixed determinant:

Let Q be a positive definite $\mathbb{Z}$-valued quadratic form in n variables. We say that two quadratic forms are equivalent over a ring R if one can obtained from the other by an invertible linear change of variables over R, and write $Q \sim_R Q'$.

We denote the $\mathbb{Z}$-equivalence class of Q by $[Q]$, and define the genus of Q, denoted $Gen(Q)$, as all quadratic forms Q' s.t. $Q' \sim_{\mathbb{Z}_p} Q$ for all primes p and $Q' \sim_{\mathbb{R}} Q$.

We define the mass of Q by

$$Mass(Q) := \sum_{[Q] \in Gen(Q)} \frac{1}{\# Aut(Q)}$$

which is known to be a finite sum of rational #'s.

Facts about the mass:

- Mass(Q) is a genus invariant that can be computed as an infinite product of local densities

$$\mathrm{Mass}(Q) = 2 \cdot \prod_v \beta_v(Q)^{-1}$$

where

$$\beta_v(Q) := \frac{1}{2} \lim_{\substack{U \supseteq \{Q\} \\ \text{open} \\ U \to \{Q\} \\ U \subseteq M_n(\mathbb{Z}_v)}} \frac{\mathrm{vol}(\tilde{Q}^{-1}(U))}{\mathrm{vol}(U)}$$

and $Q \longleftrightarrow A \in \mathrm{Sym}_n(\mathbb{Z})$ defines a map

$$\begin{aligned} M_n(\mathbb{Z}_v) &\xrightarrow{\tilde{Q}} \mathrm{Sym}_n(\mathbb{Z}_v) \\ X &\longmapsto {}^tXAX. \end{aligned}$$

When $v = p$ prime, we can write $\beta_p(Q)$ as

$$\beta_p(Q) = \frac{1}{2} \lim_{\alpha \to \infty} \frac{\#\{X \in M_n(\mathbb{Z}/p^\alpha\mathbb{Z}) \mid {}^tXAX = A\}}{p^{\frac{n(n-1)}{2} \cdot \alpha}}.$$

- There are explicit local formulas for Mass(Q), though the literature has room for improvement.

The Conway-Sloane explicit mass formula:

Suppose Q is pos. def. $\mathbb{Z}$-valued in $n \geq 3$ variables.

$$\mathrm{Mass}(Q) = 2 \cdot \pi^{-\frac{n(n+1)}{4}} \left[\prod_{j=1}^{n} \Gamma(j/2)\right] \left[\prod_{p} 2 m_p(Q)\right]$$

where the p-masses $m_p(Q)$ are defined by

$$m_p(Q) := \underbrace{\left[\prod_{q \in p^{\mathbb{Z}}} M_p(Q_p)\right]}_{\text{"Diagonal product"}} \cdot \underbrace{\left[\prod_{\substack{q, q' \in p^{\mathbb{Z}} \\ q < q'}} \left(q'/q\right)^{\frac{n(q)\, n(q')}{2}}\right]}_{\text{"Cross product"}} \cdot \underbrace{2^{n(I,I) - n(II)}}_{\text{"Type factor"}}$$

where $M_p(Q_p)$ is an explicit local Euler factor of a product of ζ and L-functions, the $n(q)$ are the Jordan dimensions of scale q, and $n(I,I)$ and $n(II)$ come from the Jordan splitting at $p = 2$.

What is the mass good for?

- Can study the # of classes in $\mathrm{Gen}(Q)$ locally (given bounds for sizes of automorphism groups).
- Enumeration of classes in a given Genus $\mathrm{Gen}(Q)$.

In this talk we will be interested in the <u>total mass</u> of given (Hessian) determinant D for positive definite quadratic forms in n variables, defined by

$$\mathrm{TMass}_n(D) := \sum_{\substack{[Q] \text{ with} \\ \det_H(Q) = D \\ \dim(Q) = n}} \frac{1}{\#\mathrm{Aut}(Q)}$$

$$= \sum_{\substack{\text{Genera } G \text{ with} \\ \det_H(G) = D \\ \dim(G) = n}} \mathrm{Mass}(G).$$

<u>Main Questions:</u>

- How can we understand $\mathrm{TMass}_n(D)$?
- Does this have any natural structure?
- How does $\mathrm{TMass}_n(D)$ behave as $D \to \infty$?

(<u>Remark:</u> The total mass can also be defined over any # field F by additionally specifying a signature σ.)

Why study the total mass?

- It's a coarser invariant than Mass(Q), so it might be simpler to understand.
- It is related to the average size of the 2-torsion in class groups of n-monogenized cubic rings.
- It generalizes the Hurwitz class numbers for positive definite binary quadratic forms (which give an interesting weight $3/2$ modular form).
- It is related to certain Shintani Zeta functions.

To understand the total mass it is useful to adopt the language of formal series.

We define the formal Dirichlet series

$$D_{TMass,n}(s) := \sum_{D \in \mathbb{N}} TMass_n(D) \cdot D^{-s}.$$

<u>Main Question:</u> What can we say about the structure of $D_{TMass,n}(s)$?

(H.) <u>Result #1:</u> If $n \geq 3$ is odd, then

$$D_{TMass,n}(s) = K_n \left[D_{A,n}(s) + D_{B,n}(s) \right]$$

some explicit constant

Eulerian Dirichlet series

where K_n is some explicit constant and both Dirichlet series $D_{A,n}(s)$ and $D_{B,n}(s)$ are given as Euler products.

(<u>Remark:</u> When n is even a similar, but more complicated result holds.)

<u>Strategy of Proof:</u> Sum up the masses of all genera of determinant D, as locally as possible.

$$\mathrm{TMass}_n(D) = \sum_{\substack{\text{Genera } G \\ \det(G)=D \\ \dim(G)=n}} 2 \cdot \prod_v \beta_v^{-1}(G)$$

$$= \underbrace{\left[2 \cdot \prod_v \beta_{v,n}^{-1}(\tilde{D})\right]}_{\mathcal{K}_n(D)} \cdot \sum_G \prod_v \frac{\beta_{v,n}(\tilde{D})}{\beta_v(G)}$$

$$= \left[2\,\mathcal{K}_n(D)\right] \cdot \sum_G \prod_{\underbrace{p \mid 2D}_{\text{call this } \Pi}} \frac{\beta_{p,n}(\tilde{D})}{\beta_p(G)}$$

Look at these genera as tuples of local genera.

$\cancel{= 2\,\mathcal{K}_n(\tilde{D})}$

$$= 2\,\mathcal{K}_n(D) \cdot \sum_{\substack{(\varepsilon_p)_{p\in\Pi} \subseteq \{\pm1\}^{|\Pi|} \\ \prod_p \varepsilon_p = 1}} \sum_{\substack{\text{Tuples } (G_p)_{p\in\Pi} \text{ with} \\ c_p(G_p)=\varepsilon_p \\ \det(G_p)=D}} \left(\prod_{p\in\Pi} \frac{\beta_{p,n}(\tilde{D})}{\beta_p(G_p)}\right)$$

$$= 2\,\mathcal{K}_n(D) \cdot \sum_{\substack{(\varepsilon_p)_{p\in\Pi} \\ \prod_p \varepsilon_p = 1}} \prod_{p\in\Pi} \underbrace{\sum_{\substack{\text{Tuples}(G_p) \\ c_p(G_p)=\varepsilon_p \\ \det(G_p)=D}} \frac{\beta_{p,n}(\tilde{D})}{\beta_p(G_p)}}_{\text{Call this } M_{p,n}^{\varepsilon}(D_p)}$$

We now isolate the dependence on ε_p by defining

$$A_{p,n}(D_p) := M^{+}_{p,n}(D_p) + M^{-}_{p,n}(D_p)$$

$$B_{p,n}(D_p) := M^{+}_{p,n}(D_p) - M^{-}_{p,n}(D_p)$$

$$\Rightarrow \boxed{M^{\varepsilon_p}_{p,n}(D_p) = \frac{A_{p,n}(D_p) + \varepsilon_p B_{p,n}(D_p)}{2}}$$

$$\Rightarrow TMass_n(D) = 2\mathcal{K}_n(D) \sum_{\substack{(\varepsilon_p)_{p\in \mathbb{T}} \\ \prod \varepsilon_p = 1}} \prod_{p \in \mathbb{T}} \frac{A_{p,n}(D) + \varepsilon_p B_{p,n}(D)}{2}$$

<u>Useful Lemma:</u> Let μ_N be the N^{th} roots of unity, $\mathbb{T}$ = finite set, and x_i, y_i be indeterminates for all $i \in \mathbb{T}$. Then

$$\sum_{\substack{(\varepsilon_i)_{i\in\mathbb{T}} \in \mu_N^{|\mathbb{T}|} \\ \text{with } \prod \varepsilon_i = \zeta}} \prod_{i \in \mathbb{T}} (x_i + \varepsilon_i y_i) = N^{|\mathbb{T}|-1}\left[\prod_i x_i + \zeta \prod_i y_i\right]$$

$$\Rightarrow TMass_n(D) = 2\mathcal{K}_n(D) \cdot \frac{2^{|\mathbb{T}|-1}}{2^{|\mathbb{T}|}} \cdot \Big[\underbrace{\prod_{p\in\mathbb{T}} A_{p,n}(D)}_{A_n(D)} + \underbrace{\prod_{p\in\mathbb{T}} B_{p,n}(D)}_{B_n(D)}\Big]$$

$$= \mathcal{K}_n(D) \cdot \left[A_n(D) + B_n(D)\right].$$

(When n is odd $\mathcal{K}_n(D)$ is independent of D!)

Ok, but can we say anything about the Euler factors of $D_{A,n}(s)$ and $D_{B,n}(s)$?

(H.) <u>Result #2:</u> The Euler factors of $D_{A,n}(s)$ and $D_{B,n}(s)$ are rational functions in p^{-s}.

<u>Sketch of Proof:</u> For any fixed n there are finitely many "Jordan block structures" (i.e. ordered tuples of $n_i \in \mathbb{N}$ summing to n), and each gives a predictable mass contribution that varies as a rational function of p^{-s} as we vary the choice of scale of each block (coming from the varying "cross product" factor).

<u>Ex:</u> $n=3$, $p \neq 2$

$$3 \longrightarrow \overset{(\dim 3)}{L_0}$$
$$2+1 \longrightarrow \overset{(\dim 2)}{L_0} + p^{\alpha} \overset{(\dim 1)}{L_1}$$
$$1+2 \longrightarrow \overset{(\dim 1)}{L_0} + p^{\alpha} \overset{(\dim 2)}{L_1}$$
$$1+1+1 \longrightarrow L_0 + p^{\alpha} L_1 + p^{\beta} L_2$$

Ok, but how explicit can we make this?

(H.) **Result #3:** Suppose that $n = 3$. Then

$$D_{TMass,\, n=3}(s) = \frac{1}{48} \cdot 2^{-s} \left[\zeta(s-1)\,\zeta(2s-1) - \zeta(s)\,\zeta(2s-2) \right].$$

Corollary:

$$TMass_{n=3}(D) = \frac{1}{48} \sum_{\substack{D = 2ab^2 \\ a,b>0}} (ab - b^2).$$

Super-Sketch of Proof: Compute $M^{\pm}_{p,\, n=3}(D)$.

$p \neq 2 \Rightarrow$ 4 Jordan Block structures

$p = 2 \Rightarrow$ 20 "Jordan Block structures"

Final Remarks: 1) These results generalize to any # field F where $p = 2$ splits completely.

2) This is essentially an explicit evaluation of a Shintani Zeta function.

Thanks for staying,

and thanks again to

the organizers:

- Billy Chan
- Lenny Fukshansky
- Rainer Schulze-Pillot
- Jeff Vaaler

Have a safe trip home!