Mixed Boundary Value Problems in Nonsmooth Domains 11frg158

Katharine Ott (University of Kentucky),
Irina Mitrea (Temple University)

12 June 2011–19 June 2011

1 Overview of the Field

The purpose of this Focused Research Group (FRG) was to consider several open problems regarding elliptic boundary value problems in domains with nonsmooth boundaries and with mixed boundary conditions of Dirichlet and Neumann type. The questions considered were motivated in part by recent progress on the mixed problem or Zarema's problem for the Laplacian in Lipschitz domains with L^p data, $1 < p < \infty$. If Ω is a bounded Lipschitz domain in $\mathbf{R}^n$, $n \geq 2$, the mixed problem for the Laplacian reads

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ u\big|_D = f_D \in L^p_1(D), \\ \dfrac{\partial u}{\partial \nu}\Big|_N = f_N \in L^p(N), \\ (\nabla u)^* \in L^p(\partial\Omega), \end{cases} \tag{1}$$

where $\partial\Omega = D \cup N$ and D and N are disjoint, ν is the outward unit normal vector to $\partial\Omega$, and where for a given set E, $\big|_E$ denotes the restriction to E. Hereafter, for $E \subseteq \partial\Omega$ measurable, and $1 < p < \infty$, $L^p(E)$ stands for the Lebesgue space of p-integrable functions on E with respect to the surface measure on $\partial\Omega$ and $L^p_1(E)$ is the L^p-based Sobolev space of order one on E. Also, for any function $v : \Omega \to \mathbf{R}$, v^* stands for the non-tangential maximal function of v given by

$$v^*(x) := \sup_{y \in \Gamma_\kappa(x)} |v(y)|, \qquad x \in \partial\Omega, \tag{2}$$

where for $x \in \partial\Omega$ and some fixed $\kappa >> 1$, we have set

$$\Gamma_\kappa(x) := \{y \in \Omega \,:\, |x - y| \leq \kappa \mathrm{dist}(y, \partial\Omega)\} \tag{3}$$

to be the non-tangential approach region with vertex at x. The study of the mixed problem (1) in Lipschitz domains is listed as open problem 3.2.15 in Kenig's CBMS lecture notes [6].

During the past thirty years there has been a great deal of interest in the classical Dirichlet and Neumann boundary value problems for the Laplacian in domains with varying degrees of smoothness, and especially Lipschitz domains (see [2] and [5] for two fundamental papers). The Lipschitz setting is significant in terms of both applications and theory. Mixed boundary value problems naturally model the behavior of several physical quantities arising in the modeling of heat transfer, metallurgical melting, stamp problems in elasticity, wave phenomena, etc., and since nonsmooth regions arise in nature, we want to understand how the geometry of a region is related to the solution of boundary value problems posed on the region. From a theoretical standpoint, the equations under consideration are constant coefficient, homogeneous differential operators and the family of solutions to such an operator is preserved by the dilations $x \to rx$ on $\mathbf{R}^n$. The class of Lipschitz domains is also preserved by these dilations and includes domains with interesting features at all length scales. Thus, the analysis of boundary value problems in Lipschitz domains is a natural area of study and it involves fundamentally new problems as compared to smoother domains.

Under appropriate conditions on D and N, recent work of J. Taylor and FRG participants K. Ott and R. Brown [8, 9] shows that there exists $p_0 > 1$, with p_0 depending on the Lipschitz constant of the domain and the dimension n, so that the boundary value problem (1) is well-posed for $p \in (1, p_0)$. In other words, it has been proved that (1) has a solution and this solution is unique in the class of functions satisfying $(\nabla u)^* \in L^p(\partial\Omega)$. In the case $p = 1$, the authors prove results for the mixed problem with data from Hardy spaces. Except for the exact value of p_0, simple examples show that this is the best possible result. Even in a smooth domain, we are not able to solve (1) in the case $p = 2$.

2 Outcome of the Meeting

2.1 The mixed problem for elliptic equations in Lipschitz domains with L^p data

The first open problem that the participants addressed was to find appropriate conditions on the domain, the boundary, and the data, which guarantee that the gradient of the solution of

$$\begin{cases} \mathcal{L}u = 0 \quad \text{in } \Omega, \\ u\,|_{D} = f_D \in L^p_1(D), \\ \dfrac{\partial u}{\partial \nu}\Big|_N = f_N \in L^p(N), \\ (\nabla u)^* \in L^p(\partial\Omega), \end{cases} \tag{4}$$

lies in $L^p(\partial\Omega)$ for some $1 \leq p < \infty$. Here $\mathcal{L}$ is a second order elliptic differential operator with constant coefficients and $\frac{\partial}{\partial \nu}$ generically denotes a conormal derivative associated with $\mathcal{L}$. When $\mathcal{L} = \Delta$ then (4) becomes (1) and in the latter situation, J. Taylor, K. Ott, and R. Brown established L^p estimates for the solution of (1) as discussed in the previous section. A key ingredient of their proof is Hölder estimates at the boundary for the Green function associated to the mixed boundary value problem. Obtaining these estimates when $\mathcal{L}$ is a matrix-valued differential operator (as in the case of systems), rather than a scalar operator, is a challenge and a problem that the participants confronted during the stay in Banff.

As a first step in approaching this problem, FRG participant S. Kim outlined his recent work on the Green functions for boundary value problems for elliptic systems. Kim had not previously considered mixed boundary conditions, but his work with collaborators [3, 4] seems applicable. The methods presented by Kim will provide a new and simpler approach to the Green function estimates used in the work of Taylor, Ott and Brown. We are currently working to use these techniques to prove the well-posedness of the mixed problem for the Lamé system of elasticity in a large class of two-dimensional Lipschitz domains with data in L^p for p near 1.

2.2 Scattering for the Helmholtz equations

In this research direction, FRG participant F. Reitich discussed the problem of scattering for the Helmholtz equation in the complement of a compact obstacle, $\bar{\Omega}$, where we impose Dirichlet boundary conditions on part of the boundary of the obstacle and Neumann boundary conditions on the remainder of the boundary.

Thus, for $k \in \mathbf{R}^n$, we are interested in the boundary value problem

$$\begin{cases} (\Delta + |k|^2)v = 0 \quad \text{in } \mathbf{R}^n \setminus \bar{\Omega}, \\ v\big|_D = e^{ik\cdot x}, \\ \frac{\partial v}{\partial \nu}\big|_N = \frac{\partial}{\partial \nu} e^{ik\cdot\cdot}, \\ v \text{ satisfies the outgoing radiation condition.} \end{cases} \tag{5}$$

Unpublished computational experiments for this problem motivated several questions which were addressed by the participants.

To begin, there are several boundary integral formulations of the scattering problem (5) that are used for computations. I. Mitrea and K. Ott [7] have successfully employed Mellin analysis techniques for the treatment of transmission boundary value problems for second order elliptic partial differential equations. These techniques have also been applied to the treatment of (1) in the case where Ω is a polygon and the solution is given by a single layer potential representation. Other singular integral representations of the solution of (1), and in turn (5), are currently being studied.

A second question related to the scattering problem arises from the following convention. To simplify computations of solutions to the mixed problem, Reitich has found it convenient to replace the sharp discontinuity in the boundary by a smooth transition. This regularization gives a family of Robin problems depending on a parameter. Two natural questions to ask are do the solutions of these Robin problems converge to the mixed problem, and can we give estimates for the rate of convergence?

To give a concrete form of this question, consider the case of homogeneous boundary conditions,

$$\begin{cases} \Delta u = F \quad \text{in } \Omega, \\ \chi_\epsilon u + (1 - \chi_\epsilon)\frac{\partial u}{\partial \nu}) = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{6}$$

If we let $\chi_0 = \chi_D$ be the indicator function of D and χ_ϵ be a family of smooth approximations to χ_D, then a natural question to ask is if the solutions of the boundary value problem with $\epsilon > 0$ converge to the solution of the mixed problem corresponding to $\epsilon = 0$. Another interesting question is to find a similar regularization of the boundary integral equations for the mixed problem with inhomogeneous boundary conditions and to establish that solutions of the regularized problem converge to the solution of the mixed problem. The coefficient of the normal derivative for the Robin problem vanishes and such problems do not seem to be widely studied.

3 Open Problems

During our time in Banff several additional, and promising, avenues for future research were identified. Building on the recent progress of studying the mixed problem for the Lamé system of elastostatics in two-dimensional Lipschitz domains, an important open problem for future research is to consider other systems such as the stationary Stokes system of fluid flow. M. Wright has a great deal of experience with the Stokes system [1, 11]. The Focused Research Group allowed Wright to become familiar with the techniques used to attack the mixed problem and he is currently working on adapting these techniques to the study of boundary value problems with mixed boundary conditions for the Stokes system.

A second interesting problem that arose out of discussions in Banff is to find the optimal range of p values for which the mixed boundary value problem with L^p data is well-posed. This would be of interest in even when the domain is smooth. We conjecture that the optimal range is p in the interval $(1, 2)$. This question may be approached in the context of the polygonal domains where the Mellin transform methods used by FRG participants I. Mitrea and K. Ott should give a complete answer in two dimensional curvilinear polygons.

Another area that bears further investigation is the use of layer potential methods for the mixed problem. G. Verchota [10] established well-posedness for the Dirichlet and Neumann problems for the Laplacian by establishing the invertibility of classical layer potentials on the boundary. It is not known if we can attack the mixed problem by layer potentials. However, it was observed during our discussions that the well-posedness of the mixed problem implies that solutions may be represented by layer potentials. Such results are of interest as the representation provides a foundation for numerical schemes to compute solutions to the mixed problem.

4 Conclusion

This Focused Research Group brought together six mathematicians who had not previously worked together, but who shared a common interest in mixed boundary value problems. One unique aspect of this FRG is that it addressed both theoretical and numerical problems related to the topic. Several promising areas of research were identified and new collaborations were begun to attack these problems.

References

[1] R. M. Brown, I. Mitrea, M. Mitrea and M. Wright. Mixed boundary value problems for the Stokes system, *Trans. Amer. Math. Soc.*, **362** (3): 1211–1230, 2010.

[2] B. E. J. Dahlberg. Estimates of harmonic measure, *Arch. Rational Mech. Anal.*, **65** (3): 275–288, 1977.

[3] H. Dong and S. Kim. Green's matrices of second order elliptic systems with measurable coefficients in two dimensional domains, *Trans. Amer. Math. Soc.*, **361** (6): 3303–3323, 2009.

[4] S. Hoffman and S. Kim. The Green function estimates for strongly elliptic systems of second order, *Manuscripta Math.*, **124** (2): 139–172, 2007.

[5] D. S. Jerison and C. E. Kenig. The Neumann problem on Lipschitz domains, *Bull. Amer. Math. Soc.*, **4**: 203–207, 1982.

[6] C. E. Kenig. *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems*, Conference Board in Mathematical Sciences, Washington, DC, 1994.

[7] I. Mitrea and K. Ott. Counterexamples to the well-posedness of L^p transmission boundary value problems for the Laplacian, *Proc. Amer. Math. Soc.*, **135**: 2037–2043, 2007.

[8] K. A. Ott and R. M. Brown. The mixed problem for the Laplacian in Lipschitz domians, arXiv:0909.0061 [math.AP], 2009.

[9] J. L. Taylor, K. A. Ott, and R. M Brown. The mixed problem in Lipschitz domains with general decompositions of the boundary, Preprint 2011.

[10] G. C. Verchota. Layer potentials and regularity for the Dirichlet problem for Laplace's equaion on Lipschitz domians, *J. Funct. Anal.*, **59**: 572–611, 1984.

[11] M. Wright. *Boundary value problems for the Stokes system in arbitrary Lipschiz domains*, ProQuest LLC, Ann Arbor, MI, 2008.