REPORT ON BIRS WORKSHOP 10W5096 "WHITTAKER FUNCTIONS, CRYSTAL BASES, AND QUANTUM GROUPS"

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1. Objectives of the workshop

The goals of the workshop were to explore several new connections between Whittaker functions and combinatorial representation theory that have recently emerged, and to encourage collaboration between researchers in number theory, combinatorial representation theory, physics, and special functions. We hoped that the balance between the lectures and free time as well as the intimate setting of the Banff Research Station would stimulate many informal discussions and collaborations. We were delighted to see that these objectives were fulfilled.

We would like to thank the BIRS staff for their hospitality and efficiency.

2. Topics of the workshop

The Casselman-Shalika formula and its generalizations. Given a Lie group or p-adic group, and "Langlands parameters"— a conjugacy class in its Langlands dual group (a complex analytic Lie group)—one may define a spherical Whittaker function. When the group is $\mathrm{GL}(2,\mathbb{R})$ these functions are the confluent hypergeometric functions introduced by Whittaker and Watson [43], hence the name.

The Casselman-Shalika formula describes the values of a spherical p-adic Whittaker function in a surprising and beautiful way [15]. The values are simply the characters of the finite-dimensional irreducible representations of the dual group evaluated at the Langlands parameter conjugacy class. This formula has many applications in automorphic forms and number theory, being basic to both the Rankin-Selberg method and the Langlands-Shahidi method of constructing L-functions. It has also recently attracted the attention of physicists, as in the work of Gerasimov, Lebedev and Oblezin in connection with Givental's work on mirror symmetry [26, 27].

Whittaker functions on metaplectic covers have also been a subject of study since the fundamental work of Kazhdan and Patterson [34], but the problem of extending the Casselman-Shalika formula to metaplectic groups was not considered much until recently due to lack of uniqueness of Whittaker models. These obstacles have been overcome by emphasizing newly discovered connections with representation theory, particularly with crystal graphs and quantum groups.

There are now two distinct types of generalizations of the Casselman-Shalika formula to metaplectic covers. To explain these, we note that by the Weyl character formula, the characters that are the special values of the original Casselman-Shalika formula are expressed as alternating sums of characters over the Weyl group. One generalization (primarily due to Chinta and Gunnells [18]) is again a sum over the Weyl group where these characters are replaced by products of Gauss sums. In the other generalization (primarily due to Brubaker, Bump, and Friedberg), the sum is similar to the realization of the character as a sum over basis vectors of a certain highest weight vector, though the monomials attached to each weight are multiplied by a function defined on the associated crystal graph. A complete understanding of these two approaches and their connection to Whittaker coefficients and to each other has only been established in type A. Partial results exist for other types, but it appears that a deeper understanding of the relationships between automorphic forms and combinatorics, particularly crystal bases, may be needed to move further.

When the metaplectic cover is trivial, these new formulas for Whittaker functions should recover the character predicted by the Casselman-Shalika formula. Remarkably, several such identities existed as purely combinatorial results in the literature. In this context, they are deformations of the Weyl character formula and the Weyl denominator formula found by Tokuyama (type A), Hamel and King (type C), Okada (type B) and others.

Crystal Bases. Crystal graphs and crystal bases were introduced by Kashiwara as combinatorial structures on representations of quantum groups. In the context of Whittaker functions, the relevant quantum group required is the quantized enveloping algebra of the Langlands dual group. Crystal bases provide a better context for an enormous amount of existing literature on tableaux, which are fundamental objects of study in combinatorics.

For examples of representations of Whittaker functions as sums over crystal bases, see Brubaker, Bump and Friedberg [11,12], Beineke, Brubaker and Frechette [2, 3], and Chinta and Gunnells [17].

The construction of p-adic Whittaker coefficients via crystal graphs should have connections to Berenstein and Kazhdan's theory of geometric crystals [4, 5]. This work gives new constructions of Kashiwara crystals, including those for modules over the Langlands dual group $\hat{G}$, using so-called positive unipotent bicrystals on the open Bruhat cell Bw_0B, where w_0 denotes the longest element of the Weyl group of G. Their method uses tropical geometry to make the transfer between the two types of crystals. Given that the p-adic integral defining the Whittaker coefficient is supported on the big Bruhat cell, and that the ingredients used to construct it have such strong resemblance to the ingredients and properties of objects in the theory of geometric crystals, we expect further investigation of these connections will lead to a better understanding of the ways in which finite dimensional representations appear in automorphic forms.

Kac-Moody developments. Since crystal bases and quantum groups can be defined in the general context of Kac-Moody Lie algebras, it becomes natural to ask whether the theory described above has generalizations to affine and other infinite root systems. This is being investigated, and there is some reason to hope for an affirmative answer. Two promising developments along these lines are the work of Garland on Eisenstein series on loop groups [21–24] and of Braverman and Kazhdan on spherical Hecke algebras on affine Kac-Moody groups over local non-archimedean fields.

However another potential approach can be imagined. The theory of metaplectic Whittaker functions can be developed independently of its origin in the representation theory of p-adic groups as the study of a class of multiple Dirichlet series with functional equations. In this approach, one uses the combinatorics of crystal graphs to define the coefficients of the Dirchlet series. A prototype was work of Bucur and Diaconu [13] (in the function field case) in which the group of functional equations was the affine Weyl group $D_4^{(1)}$. This Dirichlet series was selected due to an arithmetic application to fourth moments of L-functions, but for this proposal the relevance of their work is that it gives hope that one might be able to avoid technical complications by not basing the foundations on the representation theory mentioned above.

Another prototype of this question would be to express a deformation of the Jacobi triple product as a sum over an infinite crystal for the quantum group $A_1^{(1)}$ (in Kac's notation). Preliminary investigations suggest that this will be possible.

Real groups and further deformations. Stade developed important integral representations of Whittaker functions for $\mathrm{GL}(n, \mathbb{R})$ that allow one to express certain integrals as products of Gamma functions. These have applications in the Rankin-Selberg method. Oda, Hirano, and Ishii have developed formulas for archimedean Whittaker functions that might have connections with the p-adic theory, since for Sp(6) their formula is a sum over Gelfand-Tsetlin patterns, which are in bijection with elements of crystal bases for GL(3). Stade's work, as well as this work of Oda, Hirano, and Ishii, are points where there are strong analogies between the real and p-adic theories.

Furthermore, the physicists Gerasimov, Lebedev and Oblezin's obtained a deformation of the Casselman-Shalika formula [26, 27] in connection with Givental's work on mirror symmetry, which has been recently generalized beyond type A by Cherednik [16]. Strikingly, their theory simultaneously connects both the real and p-adic Whittaker functions: their deformations have two parameters q and t, and taking $q \to 1$ gives the classical Whittaker function, while $q \to 0$ gives the p-adic Whittaker function. This may be related to the fact that the Macdonald polynomials interpolate between archimedean and p-adic spherical functions [38], which is a point of contact with the theory of double affine Hecke operators, that was discussed by two speakers at the conference.

Statistical Mechanical Interpretations. It has been realized that Whittaker functions may be realized as partition functions of statistical-mechanical systems. These are exactly-solved lattice models, amenable to the methods of Baxter [1]. There is a potential point of contact here with the work of Gerasimov, Lebedev and Oblezin [25], for their emphasis on the Baxter Q-operator in the connection with their investigations of Whittaker functions, both real and archimedean. It is connected with both the crystal base description, since crystals may be used to parametrize the states of the physical system, and with the method of Chinta and Gunnells [18], since the functional equations that are the basis of their work express the commutativity of transfer matrices, which may be studied using the Yang-Baxter equation. Finally, it is well-known in physics that two-dimensional statistical mechanical models are equivalent to one-dimensional quantum mechanical models such as spin chains, and so there is a large amount of potentially relevant mathematical physics. See Brubaker, Bump, Chinta, Friedberg and Gunnells [8], Brubaker, Bump and Friedberg [10], Brubaker, Bump, Chinta and Gunnells [9], Hamel and King [28], and Ivanov [32].

3. Contents of the talks

The talks at the workshop reported on many new results related to the above list of topics. There were also some expository talks from experts, on both the automorphic and representation theory sides, designed to help build bridges between the different topics. Finally, there was an evening problem session that both summarized open questions mentioned in the talks and also generated new ideas. In the following we summarize the contents of each talk. Junior speakers are indicated by a bullet (•).

Arkady Berenstein (University of Oregon) spoke on *From geometric crystals to crystal bases.* The goal of his talk was to construct crystal bases (for irreducible modules over semisimple Lie algebras) by means of geometric crystals. Geometric and unipotent crystals were introduced a few years ago in a joint work with David Kazhdan as a useful geometric analogue of Kashiwara crystals [4, 5]. More recent observations (based on his recent joint paper with David Kazhdan) show that geometric crystals, in addition to providing families of piecewise-linear parametrizations of crystal bases, also reveal such hidden combinatorial structures as 'crystal multiplication' and 'central charge' on tensor products of crystal bases.

Thomas Bliem (•) (San Francisco State University) spoke on *Expected degree of weights in Demazure modules of* $\widehat{\mathfrak{sl}}_2$. He reported on a recent result [7], joint wiht S. Kousidis, about the characters of Demazure modules for the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. Namely, they calculate the derivative of these characters at $h = 0$. One can immediately reduce this to only considering the "basic specialization" of the character, i.e., the generating function of the dimensions of the eigenspaces for a scaling element d. In this language, they compute the derivative of this function at $q = 1$. For the proof, they still use another language and say that we compute the expected value of the degree distribution. En passant they obtain a new proof of Sanderson's dimension formula for these Demazure modules.

Ben Brubaker (MIT) spoke on *Modeling p-adic Whittaker functions I.* This talk, which was expository, introduced the themes of the workshop through a discussion of p-adic Whittaker functions for unramified principal series. He mentioned known methods for giving explicit descriptions of these Whittaker functions, including new expressions as generating functions on crystal bases and other combinatorially defined data associated to bases for highest weight representations.

Daniel Bump (Stanford University) presented *Modeling p-adic Whittaker functions II*, which was a continuation of Brubaker's talk. This talk looked deeper into representations of p-adic Whittaker functions. For spherical Whittaker functions on an algebraic group, these are the same as the characters of the irreducible characters of the L-group, but the speaker was also interested in metaplectic covers of the algebraic group, in which case these are similar to but different from such characters. The speaker discussed representations coming from the six-vertex model in statistical physics and explained their relationship to the crystal base description.

Both parts of this talk were videotaped.

Adrian Diaconu (University of Minnesota) gave the talk *Trace formulas and multiple Dirichlet series.* He discussed the multiple Dirichlet series attached to the problem of computing higher moments of quadratic L-functions [13]. Such series are analogues of Weyl group multiple Dirichlet series, but correspond to general Kac–Moody lie algebras. The speaker explained why results connecting character sums to automorphic forms indicate new complications in constructing the correct multiple Dirichlet series.

Solomon Friedberg (Boston College) spoke on *Global objects attached to p-adic Whittaker functions.* He described some of the connections between p-adic Whittaker functions and global objects, and their implications for number theory.

Angèle Hamel (Wilfried Laurier University) presented the talk *Bijective Proofs of Schur Function and Symplectic Schur Function Identities.* She gave modified jeu de taquin proofs of two symmetric function identities. The first relates shifted $GL(n)$-standard tableaux to the product of a Schur function and $\prod_{i<j}(x_i + y_j)$. This result generalizes the work of Robbins and Rumsey, Tokuyama, Okada, and others. The second identity is a symplectic character identity relating the sum of a product of symplectic Schur functions to the product $\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i+x_i^{-1}+y_j+y_j^{-1})$. This result has its origin in work of Hasegawa, King, and Jimbo and Miwa. It has previously been proved by Terada and Bump and Gamburd. This is joint work of Hamel with Ron King [29].

Joel Kamnitzer (University of Toronto) spoke on *Mirkovic-Vilonen cycles and MV basis.* Mirkovic-Vilonen cycles are a family of subvarieties of the affine Grassmannian, which under the geometric Satake correspondence give a basis for representations of reductive groups. The speaker began with older work giving a description of MV cycles using MV polytopes [33]. Then he explained more recent results, joint with Pierre Baumann, on properties of the resulting MV basis.

Alex Kontorovich (•) (Brown University) presented *Sieving in groups.* He discussed recent progress on the Affine Sieve, which aims to find primes or almost-primes in sets of integers generated by group actions. Applications include the Apollonian circle packing and prime entries in matrix groups. Portions of his talk were joint with Hee Oh, Jean Bourgain, and Peter Sarnak [36].

Kyu-Hwan Lee (•) (UConn) spoke on *Representation theory of p-adic groups and canonical bases.* In his talk, he interpreted the Gindikin-Karpelevich formula and the Casselman-Shalika formula as sums over Lusztig's canonical bases, generalizing the results of Bump-Nakasuji and Tokuyama to arbitrary split reductive groups. He also showed how to rewrite formulas for spherical vectors and zonal spherical functions in terms of canonical bases [35].

Peter McNamara (•) (MIT) gave the talk *Crystals and Metaplectic Whittaker Functions.* He studied Whittaker functions on nonlinear coverings of simple algebraic groups over a non-archimedean local field, and produced a recipe for expressing such a Whittaker function as a weighted sum over a crystal graph. In type A, he showed that these expressions agree with known formulae for the prime power supported coefficients of multiple Dirichlet series [39].

Ivan Mirkovic (UMass-Amherst) spoke on *Lusztig's Conjecture for Lie algebras in positive characteristic*, which is joint work with Bezrukavnikov and Rumynin. Their method is to reformulate a certain representation-theoretic problem in positive characteristic in terms of D-modules on flag varieties, coherent sheaves on the cotangent bundle of the flag variety and perverse sheaves on affine flag variety. His talk concluded with the remark that the same techniques should apply to quantum groups at roots of unity, but some parts of this investigation have not been worked out [6].

Maki Nakasuji (•) (Stanford University) talked on *Casselman's basis of Iwahori vectors and the Bruhat order*, which was joint work with Daniel Bump. Casselman defined a basis of the vectors in a spherical representation of a reductive p-adic group which is defined as being dual to the intertwining operators. They studied the explicit expression of this basis and obtained a conjecture, which is a generalization of the formula of Gindikin and Karpelevich. In this talk, she presented this conjecture and gave partial results using Hecke algebra with some examples and related combinatorial conjectures [14].

Sergey Oblezin (ITEP) *Whittaker functions and topological field theories* This talk was a survey of his recent results (in collaboration with A.Gerasimov and D.Lebedev) on $GL(N,\mathbb{R})$-Whittaker function and their q-deformations [26, 27] In the first part of the talk he constructed an element $Q(g)$ of spherical Hecke algebra $H(G,K)$ with $G = GL(N,\mathbb{R})$ and $K = SO(N)$, acting in the space of K-invariant functions on G. Then the Whittaker function is an eigenfunction of $Q(g)$, with the eigenvalue given by a product of Gamma-functions. Actually, the eigenvalue of the Whittaker function can be identified with the Archimedean L-function $L(s,\mathbb{C}^N)$.

Next he explained how the Archimedean L-function can be interpreted as correlation functions in (a pair of mirror dual) topological sigma-models on two-dimensional disk. In particular, the Archimedean L-function was identified with an equivariant symplectic volume of the space of maps of a disk into a complex space $\mathbb{C}^N$ with certain boundary conditions.

In the second part of the talk he defined a q-deformed $GL(N,\mathbb{R})$-Whittaker function and introduced a pair of its integral representations. He showed that the q-deformed Whittaker function coincides with a character of a Demazure module of affine Lie algebra $\hat{gl}(N)$. This result can be interpreted as an (Archimedean) q-version of the Casselman-Shalika formula for p-adic Whittaker function. Then he explained an interpretation of q-deformed Whittaker functions in terms of the spaces of maps of projective line into (partial) flag varieties.

The talk concluded by outlining directions of further research and generalizations to other Lie algebras. This was the second lecture that was videotaped.

Omer Offen (•) (Technion) spoke on *Spherical Whittaker functions on metaplectic GL(r)*. He proved a formula for a basis of spherical Whittaker functions with a fixed Hecke eigenvalue of the n-fold metaplectic cover of $GL(r)$. The formula expresses the Whittaker function as a sum over the Weyl group. He then showed that the p-part of the Weyl group multiple Dirichlet series of type A constructed by Chinta-Gunnells is expressed in terms of such a spherical Whittaker function. The computation adapts the method of Casselman and Shalika to the case that multiplicity is finite but not one. In the case of $n = 1$ the speaker recovers the Shintani, Casselman-Shalika formula. This was joint work with G. Chinta [40].

Soichi Okada (Nagoya University) gave the talk *Symmetric functions and spinor representations*. Symmetric functions are useful to the representation theory of classical groups. In this talk, the speaker introduced a family of symmetric functions with coefficients in the ring of integers adjoining a new element e with the property $e^2 = 1$, and investigated their properties. These symmetric functions can be used to describe the structure of the representation ring involving spinor representations of the Pin groups [31].

Manish Patnaik (•) (Harvard University) presented *Hecke algebras for p-adic loop groups*. He explained how to make sense of convolution algebras of double cosets in p-adic loop groups. In the spherical case, there is a Satake isomorhism which identifies this algebra explicitly. Moreover, he described an explicit form of this isomorphism (due in the classical case to Langlands and Macdonald) and its connection to the affine Gindikin-Karpelevic formula. He also explained an "Iwahoric" version of this construction. This is joint work with Gaitsgory and Kazhdan [42].

Samuel Patterson (Göttingen) spoke on *Some challenges from number theory.* One of the major applications of the ideas discussed at this conference was to the distribution of the values of Gauss sums, that is, to the circle of ideas around the Kummer Conjecture. This application is by way of the theory of metaplectic groups and forms. Although much has already been achieved, and the necessary tools exist, the speaker pointed out that there is still unfinished business. In particular one has to move from the local theory to the framework of Dedekind rings. In the talk he described the approach of Chinta and Gunnells [18] and attempted to clarify what still needs to be done.

Arun Ram (University of Melbourne) spoke on *Combinatorics and spherical functions.* This talk gave a dictionary between the affine Hecke language and the Whittaker function language. He defined the affine Hecke algebra, stated the affine Hecke algebra version of the Gindikin-Karpilevic formula, and discussed the Casselman-Shalika formula [15] from the point of view of Lusztig's 1981 paper on q-weight multiplicities [37] and from the point of view of Macdonald polynomials [38]. At the end of the talk he explained why Whittaker functions and their relatives have natural expressions with terms indexed by paths in the path model (crystal).

Siddhartha Sahi (Rutgers University) presented *An introduction to Double affine Hecke algebras.* Double affine Hecke algebras were introduced by Cherednik who used them to prove the Macdonald cojectures on root systems. In this talk Sahi provided an introduction to double affine Hecke algebras following his joint work with Bogdan Ion [30].

Gordan Savin (University of Utah) gave the talk *Two Bernstein components for the metaplectic group.* The Weil representation decomposes as a sum of two irreducible representations, odd and even Weil representations. Let $M(e)$ and $M(o)$ be the components, in the sense of Bernstein, of the category of smooth representations of the metaplectic group $Mp(2n)$ containing the even and the odd Weil representation, respectively. Let V^+ and V^- be two orthogonal p-adic spaces of dimension $2n+1$, with the trivial and non-trivial Hasse invariant, respectively. Let $B(+)$ and $B(-)$ be the Bernstein components of $\mathrm{SO}(V^+)$ and $\mathrm{SO}(V^-)$ containing the trivial representation. He described canonical equivalences of $M(e)$ and $B(+)$ and of $M(o)$ and $B(-)$. This was joint work with Wee Teck Gan.

Anne Schilling (UC Davis) spoke on *Combinatorics of Kirillov-Reshetikhin crystals.* She reviewed recent work with several coauthors (Masato Okado, Ghislain Fourier, Brant Jones) on combinatorial models for Kirillov-Reshetikhin crystals [19, 20, 41]. These are affine finite-dimensional crystals that play an important role in mathematical physics and representation theory. The affine structure makes it possible to define an energy statistics that can be used to define partition functions. At the end of her talk, she explained how the Kirillov-Reshetikhin crystals can be used to find expression for fusion/quantum cohomology coefficients.

4. Feedback and response from the conference

The workshop was well-attended by researchers at all points in their careers, and from countries all around the world, including Canada, USA, Israel, Russia, Japan, Germany, and Australia. After the completion of the workshop, the organizers received many positive comments from both senior and junior participants. One junior participant wrote *"It was great to learn about new connections between Whittaker functions (which I feel like I know pretty well) and exotic objects from representation theory."* A senior participant wrote *"The range of perspectives about the subject of the conference—automorphic, representation theoretic, combinatorial, quantum, analytic—made this an especially exciting and valuable meeting. The organizers are to be congratulated for their excellent work."* Another senior participant wrote *"This workshop really opened my eyes to a lot of beautiful stuff. I feel like my research has been jump-started."* One senior participant wrote *"Two research projects have grown directly out of ideas I gleaned from some of the talks, and/or from discussions I had in Banff with other attendees. I appreciate that opportunity."*

Based on these comments, we believe that the conference was successful. We also believe that by bringing together people from different fields with different strengths and interests, we have faciliated new collaborations. Because of the success of the workshop, we hope to apply to Banff in the future for another week, perhaps with the intent of running in 2012 or 2013, so that participants can give updates and progress reports, and so that new junior people can be introduced to this material.

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