Symmetric Functions and Spinor Representations

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Symmetric function techniques and Young diagrammatic method can be used for manipulation of characters of classical groups (over $\mathbb{C}$). For example, we are able to provide an algorithm for computing tensor product multiplicities and to show the stability for them. The theory has been developed by D. Littlewood, R. King, K. Koike, I. Terada, $\cdots$.

The aim of this talk is to give such a framework for spinor representations of $\mathbf{Pin}_N$.

Schur functions
and
Representation of GL_n

Symmetric functions

Let Λ be the ring of symmetric functions in $X = (x_1, x_2, \cdots)$:

$$\Lambda = \varprojlim_n \mathbb{Z}[x_1, \cdots, x_n]^{\mathfrak{S}_n},$$

where the project limit is taken in the category of graded rings.

Let h_k and e_k be the kth complete and elementary symmetric functions respectively :

$$h_k = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k},$$

$$e_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Then we have

$$\Lambda = \mathbb{Z}[h_1, h_2, \cdots] = \mathbb{Z}[e_1, e_2, \cdots].$$

Schur functions

For a partition λ, we define a Schur function s_λ by

$$s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1 \le i,j \le r},$$

where $r \ge l(\lambda)$ and $h_k = 0$ for $k < 0$.

The Schur function is also defined as the limit of

$$s_\lambda(x_1, \cdots, x_n) = \frac{\det\left(x_i^{\lambda_j + n - j}\right)_{1 \le i,j \le n}}{\det\left(x_i^{n-j}\right)_{1 \le i,j \le n}}.$$

Properties of Schur functions

- $\{s_\lambda\}_\lambda$ forms a basis of Λ.
- Dual Jacobi–Trudi identity :

$$s_\lambda = \det \left(e_{{}^t\lambda_i - i + j} \right)_{1 \le i,j \le r},$$

where ${}^t\lambda$ is the conjugate partition of λ and $r \ge l(\lambda)$.

- Cauchy identity :

$$\sum_\lambda s_\lambda(X) s_\lambda(Y) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

- Duality : Let ω be the involution of Λ defined by $\omega(h_k) = e_k$ ($k \ge 0$). Then we have

$$\omega(s_\lambda) = s_{{}^t\lambda}.$$

Representations of GL_n

Let $\mathrm{Rep}(\mathbf{GL}_n)$ be the representation ring of $\mathbf{GL}_n$, i.e., $\mathrm{Rep}(\mathbf{GL}_n)$ is a free $\mathbb{Z}$-module with basis consisting of irreducible characters of $\mathbf{GL}_n$, and it has a ring structure with respect to tensor products.

The irreducible polynomial representations of $\mathbf{GL}_n$ are parametrized by partitions of length $\leq n$. For such a partition λ, the corresponding irreducible character S_λ is given by

$$S_\lambda = \det \left(H_{\lambda_i - i + j} \right)_{1 \leq i,j \leq r},$$

where $r \geq l(\lambda)$ and

$$H_k = \text{character of the } k\text{th symmetric power } S^k(\mathbb{C}^n)$$
$$\text{of the vector representation of } \mathbf{GL}_n.$$

Specialization

Let $\rho_n : \Lambda \to \mathrm{Rep}(\mathbf{GL}_n)$ be the ring homomorphism given by

$$\rho_n(h_k) = H_k \quad (k \geq 0)$$

Then

$$\rho_n(e_k) = E_k = \text{character of the } k\text{th exterior power } \textstyle\bigwedge^k(\mathbb{C}^n).$$

By the dual Jacobi–Trudi identity, we have

$$\rho_n(s_\lambda) = \det\left(E_{t_{\lambda_i - i + j}}\right)$$

Hence it follows from $E_k = 0$ $(k > n)$ that

$$\rho_n(s_\lambda) = \begin{cases} S_\lambda & \text{if } l(\lambda) \leq n, \\ 0 & \text{if } l(\lambda) > n. \end{cases}$$

Tensor product

Let $\mathrm{LR}^{\lambda}_{\mu,\nu}$ be the Littlewood–Richardson coefficient :

$$s_\mu s_\nu = \sum_{\lambda} \mathrm{LR}^{\lambda}_{\mu,\nu}\, s_\lambda \quad \text{in } \Lambda$$

where λ runs over all partitions. Then by applying ρ_n, we have

$$S_\mu S_\nu = \sum_{l(\lambda)\le n} \mathrm{LR}^{\lambda}_{\mu,\nu}\, S_\lambda \quad \text{in } \mathrm{Rep}(\mathbf{GL}_n),$$

where λ runs over all partitions with length $\le n$.

Hence the decomposition of the tensor product of two irreducible polynomial representations corresponding to μ and ν is independent of n when $l(\mu) \le n$ and $l(\nu) \le n$.

Orthogonal Universal Characters
and
Representations of O_N

(Littlewood, King, Koike–Terada)

Representations of O_N

The irreducible representations of $\mathbf{O}_N$ are parametrized by partitions λ such that ${}^t\lambda_1 + {}^t\lambda_2 \leq N$. We call such a partition an N-orthogonal partition. For an N-orthogonal partition λ, the corresponding irreducible character $S_{[\lambda]}$ is given by

$$S_{[\lambda]} = \det\left(H_{\lambda_i - i + j} - H_{\lambda_i - i - j}\right)_{1 \leq i,j \leq r}$$

$$= \det\begin{pmatrix} H_{\lambda_1} - H_{\lambda_1-2} & H_{\lambda_1+1} - H_{\lambda_1-3} & H_{\lambda_1+2} - H_{\lambda_1-4} & \cdots \\ H_{\lambda_2-1} - H_{\lambda_2-3} & H_{\lambda_2} - H_{\lambda_2-4} & H_{\lambda_2+1} - H_{\lambda_2-5} & \cdots \\ H_{\lambda_3-2} - H_{\lambda_3-4} & H_{\lambda_3-1} - H_{\lambda_3-5} & H_{\lambda_3} - H_{\lambda_3-6} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $r \geq l(\lambda)$ and H_k is the character of the k-th symmetric tensor $S^k(\mathbb{C}^N)$ of the vector representation of $\mathbf{O}_N$.

Orthogonal universal characters

For any partition λ, we define a symmetric function $s_{[\lambda]}$ (called an orthogonal universal character) by

$$s_{[\lambda]} = \det\left(h_{\lambda_i-i+j} - h_{\lambda_i-i-j}\right)_{1\leq i,j\leq r}.$$

Let $\pi_N : \Lambda \to \mathrm{Rep}(\mathbf{O}_N)$ be the ring homomorphism defined by

$$\pi_N(h_k) = H_k \quad (k \geq 0).$$

Then we have

$$\pi_N(s_{[\lambda]}) = S_{[\lambda]} \quad \text{if } {}^t\lambda_1 + {}^t\lambda_2 \leq N.$$

Question : For a partition λ satisfying ${}^t\lambda_1 + {}^t\lambda_2 > N$,

$$\pi_N(s_{[\lambda]}) = ?$$

Properties of orthogonal universal characters

- Cauchy–type identity :

$$\sum_{\lambda} s_{[\lambda]}(X) s_{\lambda}(U) = \frac{\prod_{i \le j}(1 - u_i u_j)}{\prod_{i,j}(1 - x_i u_j)}.$$

- Schur function expansion :

$$s_{[\lambda]} = \sum_{\mu} \left(\sum_{\kappa} (-1)^{|\kappa|/2} \,\mathrm{LR}^{\lambda}_{\mu,\kappa} \right) s_{\mu},$$

  where κ runs over all partitions of the form $\kappa = (\alpha_1 + 1, \alpha_2 + 1, \cdots | \alpha_1, \alpha_2, \cdots)$ in the Frobenius notation.

- $\{s_{[\lambda]}\}_{\lambda}$ forms a basis of Λ.

Properties of orthogonal universal characters (cont.)

- Duality : Under the involution ω on Λ, we have

$$\omega(s_{[\lambda]}) = s_{\langle {}^t\lambda \rangle},$$

where $s_{\langle \mu \rangle}$ is the symplectic universal character given by

$$s_{\langle \mu \rangle} = \frac{1}{2} \det \left(h_{\mu_i - i + j} + h_{\mu_i - i - j + 2} \right)$$

- Dual Jacobi–Trudi type identity :

$$s_{[\lambda]} = \frac{1}{2} \det \left(e_{{}^t\lambda_i - i + j} + e_{{}^t\lambda_i - i - j + 2} \right).$$

Specialization

By the dual Jacobi–Trudi type identity, we have

$$\pi_N(s_{[\lambda]}) = \frac{1}{2}\det\left(E_{{}^t\lambda_i-i+j} + E_{{}^t\lambda_i-i-j+2}\right),$$

where

$$E_k = \text{character of the } k\text{th exterior powere } \bigwedge^k(\mathbb{C}^N).$$

This can be rewritten as

$$\pi_N(s_{[\lambda]}) = \det{}^t\!\left(\overrightarrow{E}_{\alpha_1}, \overrightarrow{E}_{\alpha_2}, \cdots, \overrightarrow{E}_{\alpha_r}\right),$$

where

$$\alpha = ({}^t\lambda_1, {}^t\lambda_2 - 1, \cdots, {}^t\lambda_r - (r-1)), \quad r = l({}^t\lambda),$$

and $\overrightarrow{E}_k$ is the row vector given by

$$\overrightarrow{E}_k = (E_k, E_{k+1} + E_{k-1}, E_{k+2} + E_{k-2}, \cdots, E_{k+(r-1)} + E_{k-(r-1)}).$$

Using this determinant expression and the relations

$$\overrightarrow{E}_k = 0 \quad \text{for } k \geq N + r,$$
$$E_N \overrightarrow{E}_k = \overrightarrow{E}_{N-k},$$

we can compute $\pi_N(s_{[\lambda]})$ if ${}^t\lambda_1 + {}^t\lambda_2 > N$.
(1) If $\alpha_i \geq N + r$ for some i, then we have

$$\pi_N(s_{[\lambda]}) = 0.$$

(2) If $\alpha_i + \alpha_j = N$ for some i and j, then we have

$$\pi_N(s_{[\lambda]}) = 0.$$

(3) Otherwise we can find a permutation $\sigma \in \mathfrak{S}_r$ and an N-orthogonal partition μ such that

$$\pi_N(s_{[\lambda]}) = \mathrm{sgn}(\sigma) S_{[\mu]}.$$

Here σ and μ are given as follows.

Let p be the index satisfying

$$\alpha_1 > \cdots > \alpha_p > \frac{N}{2} \geq \alpha_{p+1} > \cdots > \alpha_r.$$

And define a sequence β by putting

$$\beta = \begin{cases} (N-\alpha_1, \cdots, N-\alpha_p, \alpha_{p+1}, \cdots, \alpha_r) & \text{if } p \text{ is even,} \\ (N-\alpha_1, \cdots, N-\alpha_{p+1}, \alpha_{p+2}, \cdots, \alpha_r) & \\ \qquad \text{if } p \text{ is odd and } \alpha_p + \alpha_{p+1} \geq N+1, & \\ (N-\alpha_1, \cdots, N-\alpha_{p-1}, \alpha_p, \cdots, \alpha_r) & \\ \qquad \text{if } p \text{ is odd and } \alpha_p + \alpha_{p+1} \leq N-1. & \end{cases}$$

And let γ be the sequence from β obtained by rearranging in decreasing order, and σ be a permutation such that $\gamma = \sigma(\beta)$. Then the N-orthogonal partition μ is determined by the condition

$$\gamma = ({}^t\mu_1, {}^t\mu_2 - 1, \cdots, {}^t\mu_r - (r-1)).$$

Tensor product

In the ring Λ of symmetric functions, we can show that

$$s_{[\mu]}s_{[\nu]} = \sum_{\lambda} \left(\sum_{\tau,\xi,\eta} \mathrm{LR}^{\mu}_{\tau,\xi}\, \mathrm{LR}^{\nu}_{\tau,\eta}\, \mathrm{LR}^{\lambda}_{\xi,\eta} \right) s_{[\lambda]},$$

where λ and τ, ξ, η run over all partitions.

If μ and ν are N-orthogonal partitions, then we have

$$S_{[\mu]}S_{[\nu]} = \sum_{\lambda} \left(\sum_{\tau,\xi,\eta} \mathrm{LR}^{\mu}_{\tau,\xi}\, \mathrm{LR}^{\nu}_{\tau,\eta}\, \mathrm{LR}^{\lambda}_{\xi,\eta} \right) \pi_N(s_{[\lambda]}),$$

Together with the algorithm computing $\pi_N(s_{[\lambda]})$, we obtain the actual decomposition of $S_{[\mu]}S_{[\nu]}$ in the representation ring $\mathrm{Rep}(\mathbf{O}_N)$.

Stability of tensor product decomposition

Note that

$$\mathrm{LR}^{\alpha}_{\beta,\gamma} = 0$$

unless

$${}^t\beta_1 + {}^t\beta_2 + {}^t\gamma_1 + {}^t\gamma_2 \geq {}^t\alpha_1 + {}^t\alpha_2 \geq \max({}^t\beta_1 + {}^t\beta_2, {}^t\gamma_1 + {}^t\gamma_2).$$

Hence we see that, if ${}^t\mu_1 + {}^t\mu_2 + {}^t\nu_1 + {}^t\nu_2 \leq N$, then

$$S_{[\mu]} S_{[\nu]} = \sum_{\lambda} \left(\sum_{\tau,\xi,\eta} \mathrm{LR}^{\mu}_{\tau,\xi} \, \mathrm{LR}^{\nu}_{\tau,\eta} \, \mathrm{LR}^{\lambda}_{\xi,\eta} \right) S_{[\lambda]},$$

where λ runs over all N-orthogonal partitions, i.e., the decomposition is stable in N.

Spinor Universal Characters
and
Spinor Representations of Pin_N

Representations of $\mathbf{Pin}_N$

Let $\mathbf{Pin}_N$ be the pin group :

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathbf{Pin}_N \longrightarrow \mathbf{O}_N \longrightarrow 1.$$

So any representation of $\mathbf{O}_N$ can be viewed as a representation of $\mathbf{Pin}_N$. For an N-orthogonal partition λ, we denote by the same symbol $S_{[\lambda]}$ the character of the irreducible representation obtained by lifting the irreducible representation of $\mathbf{O}_N$ corresponding to λ. Similarly, let H_k and E_k denote the characters of $\mathbf{Pin}_N$ corresponding to the symmetric and exterior powers of the vector representation of $\mathbf{O}_N$.

Note that E_N is a one-dimensional character and

$$\mathbf{Spin}_N = \mathrm{Ker}\, E_N.$$

We say that an irreducible representation of $\mathbf{Pin}_N$ is

- a tensor representation if it factors through $\mathbf{O}_N$,
- a spinor representation otherwise.

We put

$$\begin{aligned} \mathrm{Rep}(\mathbf{Pin}_N) &= \text{the representation ring of } \mathbf{Pin}_N, \\ \mathrm{Rep}^+(\mathbf{Pin}_N) &= \text{span of the tensor irreducible characters}, \\ \mathrm{Rep}^-(\mathbf{Pin}_N) &= \text{span of the spinor irreducible characters}. \end{aligned}$$

Then we have

$$\mathrm{Rep}(\mathbf{Pin}_N) = \mathrm{Rep}^+(\mathbf{Pin}_N) \oplus \mathrm{Rep}^-(\mathbf{Pin}_N),$$

and

$$\mathrm{Rep}^+(\mathbf{Pin}_N) \cong \mathrm{Rep}(\mathbf{O}_N).$$

Spin representation

Let $\triangle$ be the character of the spin representation of $\mathbf{Pin}_N$, whose dimension is $2^{\lfloor N/2 \rfloor}$.

If N is odd, then

$$E_N \cdot \Delta \neq \Delta,$$

and

$$\Delta|_{\mathbf{Spin}_N} = \text{irred. character with h. w. } \left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right).$$

If N is even, then

$$E_N \cdot \Delta = \Delta,$$

and

$$\begin{aligned}\Delta|_{\mathbf{Spin}_N} = {} & \text{irred. character with h. w. } \left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{1}{2}\right) \\ & + \text{irred. character with h. w. } \left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2}\right).\end{aligned}$$

Irreducible spinor characters

Theorem 1 For a partition λ of length $\leq N/2$, we define a class function $S_{[\lambda+1/2]}$ on $\mathbf{Pin}_N$ by

$$S_{[\lambda+1/2]} = \Delta \cdot \det\left(H_{\lambda_i - i + j} - E_N H_{\lambda_i - i - j + 1}\right)_{1 \leq i,j \leq r},$$

where $r \geq l(\lambda)$. Then $S_{[\lambda+1/2]}$ is an irreducible character of $\mathbf{Pin}_N$.

Remark If N is odd, then $\mathrm{Rep}^-(\mathbf{Pin}_N)$ has a basis

$$S_{[\lambda+1/2]}, \quad E_N \cdot S_{[\lambda+1/2]} \quad (l(\lambda) \leq N/2).$$

If N is even, then $\mathrm{Rep}^-(\mathbf{Pin}_N)$ has a basis

$$S_{[\lambda+1/2]} \quad (l(\lambda) \leq N/2).$$

Idea of Proof of Theorem 1 : It is enough to show that

- $S_{[\lambda+1/2]}$ is a virtual character, i.e., an integral linear combination of characters,
- If $\langle\ \ ,\ \ \rangle$ is the canonial symmetric bilinear form on the space of class functions of $\mathbf{Pin}_N$, then

$$\langle S_{[\lambda+1/2]}, S_{[\lambda+1/2]}\rangle = 1,$$

- The value of $S_{[\lambda+1/2]}$ at the identity element of $\mathbf{Pin}_N$ is positive.

Spinor universal characters

We work in the ring $\widetilde{\Lambda}$ of symmetric functions with coefficients in the ring $\mathbb{Z}[\varepsilon]/(\varepsilon^2 - 1)$:

$$\widetilde{\Lambda} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon]/(\varepsilon^2 - 1),$$

For any partition λ, we define a symmetric function $s'_{[\lambda]}$ (called a spinor universal character) by putting

$$s'_{[\lambda]} = \det \left(h_{\lambda_i - i + j} - \varepsilon h_{\lambda_i - i - j + 1} \right)_{1 \le i,j \le r}.$$

Let $\widetilde{\pi}_N : \widetilde{\Lambda} \to \mathrm{Rep}(\mathbf{Pin}_N)$ be the ring homomorphism given by

$$\widetilde{\pi}_N(h_k) = H_k \quad (k \ge 0) \quad \text{and} \quad \widetilde{\pi}_N(\varepsilon) = E_N.$$

Then we have, for a partition λ of length $\le N/2$,

$$S_{[\lambda+1/2]} = \Delta \cdot \widetilde{\pi}_N(s'_{[\lambda]}).$$

Properties of spinor universal characters $s'_{[\lambda]}$

- Cauchy–type identity :

$$\sum_{\lambda} s'_{[\lambda]}(X) s_{\lambda}(U) = \frac{\prod_i (1 - \varepsilon u_i) \prod_{i<j} (1 - u_i u_j)}{\prod_{i,j} (1 - x_i u_j)}.$$

- Schur function expansion :

$$s'_{[\lambda]} = \sum_{\mu} \left(\sum_{\nu = {}^t\nu} (-1)^{(|\nu| + l(\nu))/2} \varepsilon^{|\nu|} \, \mathrm{LR}^{\lambda}_{\mu,\nu} \right) s_{\mu},$$

  where the inner summation is taken over all self-conjugate partitions ν.

- $\{s'_{[\lambda]}, \varepsilon s'_{[\lambda]}\}_{\lambda}$ form a $\mathbb{Z}$-basis of $\widetilde{\Lambda}$.

Properties of spinor universal characters $s'_{[\lambda]}$ (cont.)

- Duality :

$$\omega(s'_{[\lambda]}) = s'_{[{}^t\lambda]}.$$

- Dual Jacobi–Trudi type identity

$$s'_{[\lambda]} = \det\left(e_{{}^t\lambda_i - i + j} - \varepsilon e_{{}^t\lambda_i - i - j + 1}\right).$$

Specialization

By the dual Jacobi–Trudi type identity, we have

$$\widetilde{\pi}_N(s'_{[\lambda]}) = \det\left(E_{{}^t\lambda_i - i + j} - E_N E_{{}^t\lambda_i - i - j + 1}\right)_{1 \le i,j \le r},$$

where $r = l({}^t\lambda)$.

We put

$$E'_k = E_k - E_N E_{k-1},$$

and define a row vector $\overrightarrow{E}'_k$ by

$$\overrightarrow{E}'_k = \left(E'_k, E'_{k+1} + E'_{k-1}, \cdots, E'_{k+(r-1)} + E'_{k-(r-1)}\right).$$

Then the above determinant can be rewritten as

$$\widetilde{\pi}_N(s'_{[\lambda]}) = \det {}^t\!\left(\overrightarrow{E}'_{\alpha_1}, \overrightarrow{E}'_{\alpha_2}, \cdots, \overrightarrow{E}'_{\alpha_r}\right)$$

where

$$\alpha = ({}^t\lambda_1, {}^t\lambda_2 - 1, \cdots, {}^t\lambda_r - (r-1)).$$

We can use this determinant expression and the relations

$$\overrightarrow{E}'_k = 0 \quad \text{for } k \geq N + r,$$
$$\overrightarrow{E}'_k + \overrightarrow{E}'_{N+1-k} = 0$$

to compute $\widetilde{\pi}_N(s'_{[\lambda]})$ for a partition λ with length $> N/2$.
(1) If $\alpha_i \geq N + r$ for some i, then we have

$$\widetilde{\pi}_N(s'_{[\lambda]}) = 0.$$

(2) If $\alpha_i + \alpha_j = N + 1$ for some i and j, then we have

$$\widetilde{\pi}_N(s'_{[\lambda]}) = 0.$$

(3) Otherwise we can find an index p, a permutation σ and a partition μ of length $\leq N/2$ such that

$$\Delta \cdot \widetilde{\pi}_N(s'_{[\lambda]}) = (-1)^p \operatorname{sgn}(\sigma) S_{[\mu+1/2]}.$$

Here p, σ and μ are given as follows. Let p be an index such that

$$\alpha_1 > \cdots > \alpha_p > \frac{N+1}{2} \geq \alpha_{p+1} > \cdots > \alpha_r,$$

and define a new sequence β by

$$\beta = (N+1-\alpha_1, \cdots, N+1-\alpha_p, \alpha_{p+1}, \cdots, \alpha_r).$$

Let γ be the sequence obtained from β by rearranging components in decreasing order, and σ be a permutation such that $\gamma = \sigma(\beta)$. Finally a partition μ is given by

$$\gamma = ({}^t\mu_1, {}^t\mu_2, \cdots, {}^t\mu_r)$$

Example Let $\lambda = (4,3,3,3,2,2,1,1)$ and $N = 8$. Then ${}^t\lambda = (8,6,4,1)$ and

$$\alpha = (8, 6-1, 4-2, 1-3) = (8,5,2,-2).$$

There are two components larger than $(N+1)/2 = 9/2$, so $p = 2$ and

$$\beta = (9-8, 9-5, 2, -2) = (1,4,2,-2).$$

Hence

$$\gamma = (4,2,1,-2), \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

and

$${}^t\mu = (4, 2+1, 1+2, -2+3) = (4,3,3,1), \quad \mu = (4,3,3,1).$$

Hence we have

$$\Delta \cdot \widetilde{\pi}_8(s'_{[4,3,3,3,2,2,1,1]}) = (-1)^2 \cdot (-1)^2 \cdot S_{[(4,3,3,1)+1/2]}.$$

Tensor product of a spinor repr. and a tensor repr.

In order to compute the product

$$S_{[\mu+1/2]} \cdot S_{[\nu]} = \Delta \cdot \widetilde{\pi}_N(s'_{[\mu]} s_{[\nu]}) \quad \text{in } \mathrm{Rep}(\mathbf{Pin}_N),$$

it is enough to compute

$$s'_{[\mu]} \cdot s_{[\nu]} \quad \text{in } \widetilde{\Lambda}.$$

Theorem 2 In the ring $\widetilde{\Lambda}$, we have

$$s'_{[\mu]} \cdot s_{[\nu]} = \sum_{\lambda} \left(\sum_{\substack{\xi,\eta,\tau \\ \nu/\sigma\ :\ \text{v-strip}}} \mathrm{LR}^{\lambda}_{\xi,\eta} \, \mathrm{LR}^{\mu}_{\tau,\xi} \, \mathrm{LR}^{\sigma}_{\tau,\eta} \, \varepsilon^{|\nu|-|\sigma|} \right) s'_{[\lambda]},$$

where ξ, η, τ run over all partitions and σ runs over all partitions such that ν/σ is a vertical strip.

By applying the specialization $\widetilde{\pi}_N$, we obtain

$$S_{[\mu+1/2]} \cdot S_{[\nu]} = \sum_{\lambda} \left(\sum_{\xi,\eta,\tau,\sigma} \mathrm{LR}^{\lambda}_{\xi,\eta} \, \mathrm{LR}^{\mu}_{\tau,\xi} \, \mathrm{LR}^{\sigma}_{\tau,\eta} \, E_N^{|\nu|-|\sigma|} \right) \Delta \cdot \widetilde{\pi}_N(s'_{[\lambda]}),$$

where λ runs over all partitions.

If $l(\mu) + l(\nu) \leq N/2$, then we have

$$S_{[\mu+1/2]} \cdot S_{[\nu]} = \sum_{\lambda} \left(\sum_{\xi,\eta,\tau,\sigma} \mathrm{LR}^{\lambda}_{\xi,\eta} \, \mathrm{LR}^{\mu}_{\tau,\xi} \, \mathrm{LR}^{\sigma}_{\tau,\eta} \, E_N^{|\nu|-|\sigma|} \right) S_{[\lambda+1/2]},$$

where λ runs over all partitions of length $\leq N/2$. In this case, the decomposition depends only on μ and ν (and the parity of N).

Proof of Theorem 2 Consider the generating function with respect to Schur functions.

$$\sum_{\mu,\nu} s'_{[\mu]}(X) s_{[\nu]}(X) s_{\mu}(U) s_{\nu}(V)$$

$$= \frac{\prod_i (1-\varepsilon u_i) \prod_{i<j}(1-u_iu_j)}{\prod_{i,j}(1-x_iu_j)} \cdot \frac{\prod_{i\le j}(1-v_iv_j)}{\prod_{i,j}(1-x_iv_j)}$$

$$= \prod_i (1+\varepsilon v_i) \cdot \frac{1}{\prod_{i,j}(1-u_iv_j)}$$

$$\cdot \frac{\prod_i (1-\varepsilon u_i) \prod_i (1-\varepsilon v_i) \prod_{i<j}(1-u_iu_j) \prod_{i,j}(1-u_iv_j) \prod_{i<j}(1-v_iv_j)}{\prod_{i,j}(1-x_iu_j)\prod_{i,j}(1-x_iv_j)}$$

$$= \left(\sum_{k\ge 0} \varepsilon^k e_k(V)\right) \cdot \left(\sum_{\tau} s_{\tau}(U) s_{\tau}(V)\right) \cdot \left(\sum_{\lambda} s'_{[\lambda]}(X) s_{\lambda}(U\cup V)\right).$$

Now we express

$$e_k(V) \cdot s_\tau(U) s_\tau(V) \cdot s_\lambda(U \cup V)$$

as a linear combinatioin of the product of Schur functions in U and V. Finally we get

$$\sum_{\mu,\nu} s'_{[\mu]}(X) s_{[\nu]}(X) s_\mu(U) s_\nu(V)$$

$$= \sum_{\mu,\nu} \sum_{\lambda} \left(\sum_{\xi,\eta,\tau,\sigma} \varepsilon^{|\nu|-|\sigma|} \mathrm{LR}^{\lambda}_{\xi,\eta} \mathrm{LR}^{\mu}_{\tau,\xi} \mathrm{LR}^{\sigma}_{\tau,\eta} \right) s'_{[\lambda]}(X) s_\mu(U) s_\nu(V),$$

where ν runs over all partitions such that ν/σ is a vertical strip. By comparing the coefficient of $s_\mu(U)s_\nu(V)$, we obtain the desired identity.

Tensor product of two spinor repr.

We consider the product

$$S_{[\mu+1/2]} \cdot S_{[\nu+1/2]} = \Delta^2 \cdot \widetilde{\pi}_N(s'_{[\mu]} s'_{[\nu]}) \quad \text{in } \mathrm{Rep}(\mathbf{Pin}_N).$$

It is known that

$$\Delta^2 = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{r=0}^{N} E_N^r E_r & \text{if } N \text{ is odd,} \\ \displaystyle\sum_{r=0}^{N} E_N^r E_r & \text{if } N \text{ is even.} \end{cases}$$

Hence the tensor product of two spinor representations can be computed by using the following two formulae.

Theorem 3 In the ring $\widetilde{\Lambda}$, we have

$$s'_{[\mu]} \cdot s'_{[\nu]} = \sum_{\lambda} \left(\sum_{\xi,\eta,\tau} \mathrm{LR}^{\lambda}_{\xi,\eta} \, \mathrm{LR}^{\mu}_{\tau,\xi} \, \mathrm{LR}^{\nu}_{\tau,\eta} \right) s'_{[\lambda]}.$$

Also we have

$$\sum_{k \geq 0} \varepsilon^k e_k \cdot s'_{[\mu]} = \sum_{\lambda} \varepsilon^{|\lambda|-|\mu|} s_{[\lambda]},$$

where λ runs over all partitions such that λ/μ is a vertical strip.