Weak and strong limit values

Evgeny Poletsky
eapolets@syr.edu

Syracuse University
Department of Mathematics
215 Carnegie Building
Syracuse, NY 13244-1150

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Let u be a continuous exhaustion function, let $B_r = \{u < r\}$, let μ_r be measures supported by $S_r = \partial B_r$ and converging weak-$*$ to μ as $r \to \infty$ in $C^*(\overline{D})$. We are dealing with some space $\mathcal{F}^p \subset C(D)$, $p \geq 1$, which is a Banach space with the norm

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$$\|\phi\|^p_{\mathcal{F}^p} = \limsup_{r\to\infty} \int\limits_{S_r} |\phi|^p \, d\mu_r < \infty.$$

It is known that that the measures $\{\phi\mu_r\}$ and $|\phi|^p\mu_r$ converge weak-$*$ as $r \to \infty$ to measures $\nu(\phi)$ and $\nu_p(\phi)$ respectively. If $\phi \in \mathcal{F}^p_c = \mathcal{F}^p \cap C(\overline{D})$ then $\|\phi\|_{L^p(S,\mu)} = \|\phi\|_{\mathcal{F}^p}$.

Let u be a continuous exhaustion function, let $B_r = \{u < r\}$, let μ_r be measures supported by $S_r = \partial B_r$ and converging weak-$*$ to μ as $r \to \infty$ in $C^*(\overline{D})$. We are dealing with some space $\mathcal{F}^p \subset C(D)$, $p \geq 1$, which is a Banach space with the norm

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It is known that that the measures $\{\phi\mu_r\}$ and $|\phi|^p\mu_r$ converge weak-$*$ as $r \to \infty$ to measures $\nu(\phi)$ and $\nu_p(\phi)$ respectively. If $\phi \in \mathcal{F}^p_c = \mathcal{F}^p \cap C(\overline{D})$ then $\|\phi\|_{L^p(S,\mu)} = \|\phi\|_{\mathcal{F}^p}$. Also there is a kernel $P(z,\zeta)$ on $D \times S$, $S = \partial D$, such that

$$\phi(z) = \int\limits_S \phi_*(\zeta) P(z,\zeta) \, d\mu(\zeta)$$

when $\phi \in \mathcal{F}^p_c$.

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For the future we mention that in our problem there are kernels $P_r(z, \zeta)$ on $B_r \times S_r$ such that

$$\phi(z) = \int\limits_{S_r} \phi(\zeta) P_r(z, \zeta)\, d\mu_r(\zeta),\ z \in B_r$$

and for every $z \in D$ the measures $P_r(z, \cdot)\mu_r$ converge weak-$*$ to $P(z, \cdot)\mu$ as $r \to \infty$ in $C^*(\overline{D})$.

Since for every sequence $\{r_j\}$ going to ∞ the measures $\{\phi\mu_{r_j}\}$ converge weak-$*$ to a measure $\nu(\phi)$ let's look at sequences.

Let K be a compact metric space, and let $M = \{\mu_j\}$ be a sequence of regular Borel measures on K converging weak-$*$ in $C^*(K)$ to a finite measure μ. We denote the set $\operatorname{supp}\mu_j$ by K_j and $\operatorname{supp}\mu$ by K_0. Let $\phi = \{\phi_j\}$ be a sequence of Borel functions ϕ_j on K_j. We let

$$\|\phi\|_{L^p(M)} = \limsup_{j\to\infty} \|\phi_j\|_{L^p(K,\mu_j)}.$$

In general, the weak-$*$ limit $\nu(\phi$ of measures $\phi_j\mu_j$ need not to be absolutely continuous with respect to μ. But as the following lemma shows this is the case when $\|\phi\|_{L^p(K,M)} < \infty$ for some $p > 1$.

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Lemma

If $\|\phi\|_{L^p(M)} \leq A$, $p > 1$, and the measures $\phi_j\mu_j$ converge weak-$*$ to a measure $\nu(\phi)$ on K, then there is a function $\phi_* \in L^p(K_0,\mu)$ such that $\mu' = \phi_*\mu$ and $\|\phi_*\|_{L^p(K_0,\mu)} \leq A$.

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If the measures $\{\phi_j \mu_j\}$ converge weak-$*$ to a measure $\phi_* \mu$, then the function ϕ_* will be called the *weak limit values* of ϕ. We will denote by $\mathcal{A}(M)$ the space of all sequences ϕ of Borel functions ϕ_j on K_j and by $\mathcal{A}^p(M)$ those sequences which have the weak limit values and $\|\phi\|_{L^p(M)} < \infty$.

Now we know that every $\phi \in \mathcal{F}^p$, $p > 1$, has the weak limit values. Is $I(\phi) = \phi_*$?

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Theorem

If $\phi \in \mathcal{A}^p(M)$ and the measures $|\phi_j|^p \mu_j$ converge weak-$*$ to a measure ν on K, then $\nu \geq |\phi_*|^p \mu$.

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So $\|\phi\|_{\mathcal{F}^p} \geq \|\phi_*\|_{L^p(S,\mu)}$.

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We say that a sequence $\phi \in \mathcal{A}(M)$ *has the strong limit values on* K_0 *with respect to* M if there is a μ-measurable function ϕ^* on K_0 such that for any $b > a$ and any $\varepsilon, \delta > 0$ there is j_0 and an open set $O \subset K$ containing $G(a,b) = \{x \in K_0 : \ a \le \phi^*(x) < b\}$ such that

$$\mu_j(\{\phi_j < a - \varepsilon\} \cap O) + \mu_j(\{\phi_j > b + \varepsilon\} \cap O) < \delta \tag{1}$$

when $j \ge j_0$. The function ϕ^* will be called the *strong limit values* of ϕ.

The next theorem provides a convenient criterion for the existence of limit values.

Theorem

A sequence $\phi \in \mathcal{A}(M)$ has the strong limit values ϕ^* on K if and only if for every $\varepsilon, \delta > 0$ there is a function $f \in C(K)$ such that $\mu(\{|f - \phi^*| > \varepsilon\}) < \delta$ and $\mu_j(\{|f - \phi_j| > \varepsilon\}) < \delta$ for large j. Moreover, if $\|\phi\|_{L^p(M)} < \infty$ then for every $\varepsilon > 0$ the function f can be chosen so that $\|f\|_{L^p(\mu_j)} < \|\phi\|_{L^p(M)} + \varepsilon$ for large j if $p < \infty$, and $\|f\| \leq \|\phi\|_{L^\infty(M)} + \varepsilon$.

Let us indicate some properties of strong limit values.

Theorem

Suppose that ϕ has the strong limit values on K_0 equal to ϕ^*. Then:

1. *the choice of ϕ^* is unique and the sequences $c\phi$ and $|\phi|^p$ have strong limit values and $(c\phi)^* = c\phi^*$ and $(|\phi|^p)^* = |\phi^*|^p$;*

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1. *the choice of ϕ^* is unique and the sequences $c\phi$ and $|\phi|^p$ have strong limit values and $(c\phi)^* = c\phi^*$ and $(|\phi|^p)^* = |\phi^*|^p$;*
2. *if a sequence ψ has the strong limit values ψ^*, then the sequences $\phi + \psi$ and $\phi\psi$ have the strong limit values and $(\phi + \psi)^* = \phi^* + \psi^*$ and $(\phi\psi)^* = \phi^*\psi^*$.*

It is not true that the weak limit values are equal to the strong limit values even when both do exist. For example, let all μ_j be equal to the Lebesgue measure μ on $[0, 1]$. Surround the points k/j, $0 \leq k \leq j$, by intervals of equal size and of total length $1/j$. Let ϕ_j be equal j on these intervals and 0 outside. Then $\phi^* \equiv 0$, while $\phi_* \equiv 1$.

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However, under additional integrability assumptions, the strong limit values correlate in the right way with powers and products of sequences even when one of the factors has only the weak limit values as the following chain of results shows.

Theorem

Suppose that a sequence ϕ *has the strong limit values* ϕ^*.

1. *Suppose that* $\|\phi\|_{L^p(M)} = A < \infty$, $p > 1$. *Then* $\|\phi^*\|_{L^p(K,\mu)} \leq A$ *and for any* s, $1 \leq s < p$, *the measures* $|\phi|^s \mu_j$ *converge weak-*$*$ *to* $|\phi^*|^s \mu$.

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2. *Suppose that* $\|\phi\|_{L^p(M)} < \infty$, $p > 1$, *and a sequence* $\psi \in \mathcal{A}^s(M)$, *where* $s > q$ *is defined by* $1/p + 1/q = 1$. *Then the sequence* $\{\phi_j \psi_j \mu_j\}$ *converges weak-*$*$ *to* $\phi^* \psi_* \mu$.

Now we came to the main theorem.

Theorem

Let $\|\phi\|_{L^p(M)} \in \mathcal{A}^p(M)$ *for some* $p > 1$, *and the measures* $\{|\phi_j|^p \mu_j\}$ *converge weak-*$*$ *to* ν. *If*

$$\nu(K) = \int_K |\phi_*|^p \, d\mu$$

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then the sequence ϕ *has the strong limit values equal to* ϕ_**.*

We already know that always $\nu \geq |\phi_*|^p \mu$. So $\nu = |\phi_*|^p \mu$ if and only if the sequence ϕ has the strong limit values and they are equal to ϕ_* if and only if the mapping $\phi \to \phi_*$ is an isometry.

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$$\Phi_p(x) = x^p - p(x-1) - 1. \tag{2}$$

Clearly, $\Phi_p(x) \geq 0$ and $\Phi_p(x) = 0$ if and only if $x = 1$.

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Let

$$\bar{f} = ⨍_K f \, d\mu = \frac{1}{\mu(K)} \int_K f \, d\mu.$$

Replacing x with $f/\bar{f}$ in (2) and integrating both sides we get the following proposition.

Proposition

Let (K,μ) *be a measure space,* $0<\mu(K)<\infty$*,* $p\geq 1$*, and let* f *be a non-negative measurable function on* K *with* $\|f\|_{L^1(K,\mu)}<\infty$*. Then*

$$\left(\fint_K f\,d\mu\right)^{-p}\fint_K f^p\,d\mu = 1+\fint_K \Phi_p\left(\frac{f}{\overline{f}}\right)\,d\mu. \tag{3}$$

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If $p\geq 2$ *then*

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Proposition

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We use the following immediate consequence of this proposition.

Lemma

In assumptions of Proposition 14 for $p>1$ and $c>0$ there is a constant $\alpha(p,c)$ such that if the left side of (3) is smaller than $1+\varepsilon$, $\varepsilon>0$, then $\mu(\{|f/\bar f-1|>1+c\})<\alpha(p,c)\varepsilon\mu(K)$.

It became clear from the previous results that to have strong limit values the weak limit values need to exercise some control over the values of sequences. The typical form of this control is that for some kernel P

$$\phi(z) \le \int_{K_0} P(z,\zeta)\phi_*(\zeta)\,d\mu(\zeta). \tag{4}$$

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We assume that $P(z,\zeta)$ be a non-negative Borel function on $K \times K_0$ and for all j and for each fixed $z \in K_j$ the function $P(z,\zeta)$ is bounded on $K_0 = \operatorname{supp}\mu$. Let $\mathcal{A}^p(M,P)$ be the set of all sequences of Borel functions ϕ_j defined on K_j which have weak limit values ϕ_*, $\|\phi\|_{L^p(M)} < \infty$ and (4) holds for all $z \in \cup_{j=1}^{\infty} K_j$.

It is reasonable to request that the class $\mathcal{A}^p(M, P)$ contains the constants and this is equivalent to request that

$$\int_{K_0} P(z,\zeta)\, d\mu(\zeta) = 1 \tag{5}$$

for all $z \in \cup_{j=1}^{\infty} K_j$.
Now we can prove a theorem establishing the existence of strong boundary values for sequences in $\mathcal{A}^p(M, P)$.

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Now we can prove a theorem establishing the existence of strong boundary values for sequences in $\mathcal{A}^p(M, P)$.

Theorem

Suppose that the kernel $P(z,\zeta)$ satisfies (5), the functions

$$p_j(\zeta) = \int\limits_{K_j} P(z,\zeta)\, d\mu_j(z)$$

are uniformly bounded and converge weak-$*$ to 1 on K_0. Then any sequence $\{\phi_j\} \in \mathcal{A}^p(M, P)$ has strong limit values equal to ϕ_*.

So we are left with two questions about the kernel P:

1. When for $\phi \in \mathcal{F}^p$ we have

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To answer the first question observe that the inequality (4) comes from similar inequalities obtained on the interior of a domain in the following process:

Theorem

Suppose that for each j there are Borel functions $P_j(z,\zeta)$ defined on $\cup_{m=1}^{j-1} K_m \times K_j$ such that for each $z \in K_m$ the functions $P_j(z,\cdot)$, $j > m$, are uniformly bounded and have the strong limit values $P(z,\cdot)$ with respect to $M = \{\mu_j\}$.

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If $\phi \in L^p(M)$, $p > 1$, has the weak boundary values and

$$\phi_m(z) \le \int\limits_{K_j} P_j(z,\zeta)\phi_j(\zeta)\, d\mu_j(\zeta)$$

for each $z \in \cup_{m=1}^{j-1} K_m$, then $\phi \in \mathcal{A}^p(M,P)$.

The condition on the functions p_j can be obtained from the following natural phenomenon which occurs rather frequently.

Theorem

Suppose that $\mathcal{A}^p(M, P)$ *contains constants and the measures* $P(z,\zeta_j)\mu_j(z)$ *converge weak-*$*$ *to* δ_ζ *if the sequence* $\{\zeta_j\} \subset K_0$ *converges to* $\zeta \in K_0$*. Then the functions*

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are uniformly bounded and converge weak-$*$ to 1 on $\operatorname{supp}\mu$.
Moreover, if $h \in L^p(\mu)$, $p > 1$, then the sequence

$$\phi_j(z) = \int\limits_K P(z,\zeta)h(\zeta)\, d\mu(\zeta), \qquad z \in K_j,$$

belongs to $\mathcal{A}^p(M,P)$ and has weak (and, consequently, strong) limit values equal to h.

Now we go back to boundary values and our questions. Let us recall that u is a continuous exhaustion function so for all r the sets $B_r = \{u < r\}$ are connected and compactly belong to D. Suppose also that there are measures μ_r supported by $S_r = \partial B_r$ and converging weak-$*$ to μ as $r \to \infty$ and kernels $P_r(z, \zeta)$ on $B_r \times S_r$.

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We will require the kernels $P_r(z, \zeta)$ to satisfy the following conditions:

1. $$\int\limits_{S_r} P_r(z, \zeta)\, d\mu(\zeta) = 1;$$
2. for each $z \in D$ the functions $P_r(z, \zeta)$ are non-negative and uniformly bounded on S_r when r is large;

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3. for each $z \in D$ and any sequence of $r_j \to \infty$ the functions $P_{r_j}(z, \cdot)$ have the strong limit values $P(z, \cdot)$ with respect to $M = \{\mu_{r_j}\}$;

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4. the measures $P(z,\zeta_j)\mu_{r_j}(z)$ converge weak-$*$ to δ_ζ if $r_j \to \infty$ and the sequence $\{\zeta_j\} \subset \operatorname{supp}\mu$ converges to ζ.

Theorem

Let $\phi \in \mathcal{F}^p$. *Then:*

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   for $z \in D$;

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2. *if a sequence* $r_j \to \infty$ *then the sequence of functions* $\{\phi_j = \phi|_{S_{r_j}}\}$ *has the strong limit values with respect to* $\{\mu_{r_j}\}$ *equal to* ϕ_**.*
3. $\|\phi_*\|_{L^p(\mu)} \le \|\phi\|_{\mathcal{F}^p}$ *and for any* s*,* $1 \le s < p$*, the measures* $|\phi|^s \mu_r$ *converge weak-*$*$ *to* $|\phi_*|^s$ *as* $r \to \infty$*;*

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3. $\|\phi_*\|_{L^p(\mu)} \le \|\phi\|_{\mathcal{F}^p}$ *and for any s, $1 \le s < p$, the measures $|\phi|^s \mu_r$ converge weak-$*$ to $|\phi_*|^s$ as $r \to \infty$;*
4. *if $\|\psi\|_{\mathcal{F}^s} < \infty$ for a Borel function ψ on D, where $s > q$, $1/p + 1/q = 1$, and the measures $\{\psi \mu_r\}$ converge weak-$*$ to the measure $\psi_* \mu$ as $r \to \infty$, then the measures $\{\phi \psi \mu_r\}$ converge weak-$*$ to $\phi_* \psi_* \mu$ as $r \to \infty$;*

Let D be a hyperconvex domain in $\mathbb{C}^n$. This means that there is a continuous negative plurisubharmonic function u on D, called an *exhaustion function*, such that $\lim_{z\to\partial D} u(z) = 0$. This domains were studied by Demailly who proved that any bounded pseudoconvex domain with the Lipschitz boundary is hyperconvex.

Let D be a hyperconvex domain in $\mathbb{C}^n$. This means that there is a continuous negative plurisubharmonic function u on D, called an *exhaustion function*, such that $\lim_{z\to\partial D} u(z) = 0$. This domains were studied by Demailly who proved that any bounded pseudoconvex domain with the Lipschitz boundary is hyperconvex.

For such a function u let $B_u(r) = \{z \in D : u(z) < r\}$ and $S_u(r) = \{z \in D : u(z) = r\}$. Let

$$\mu_{u,r} = (dd^c u_r)^n - \chi_{D\setminus B_u(r)}(dd^c u)^n,$$

where $u_r = \max\{u, r\}$. The measure $\mu_{u,r}$ is nonnegative and supported by $S_u(r)$.

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Demailly (1985) had proved the following fundamental Lelong–Jensen formula.

Theorem

For all $r < 0$ every plurisubharmonic function ϕ on D

$$\mu_{u,r}(\phi) = \int_D \phi \mu_{u,r}$$

is finite and

$$\mu_{u,r}(\phi) - \int_{B_u(r)} \phi (dd^c u)^n = \int_{B_u(r)} (r-u) dd^c \phi \wedge (dd^c u)^{n-1}$$

$$= \int_{-\infty}^{r} dt \int_{B_u(t)} dd^c \phi \wedge (dd^c u)^{n-1} < \infty.$$

The last integral in this formula can be equal to ∞. Then the integral in the left side is equal to $-\infty$.

In the analogy to the classic case we define the Hardy spaces

$$H_u^p(D) = \{f : \|f\|_u^p = \lim_{r\to 1-} \mu_{u,r}(|f|^p) < \infty\}$$

and the Bergman spaces

$$A_{u,\alpha}^p(D) = \{f : \|f\|_{u,\alpha}^p = \int_{-\infty}^{0} |r|^\alpha e^r \mu_{u,r}(|f|^p)\, dr < \infty\}.$$

If $\mathbb{D}$ is the unit disk in $\mathbb{C}$ then the new definitions coincide with old ones if $u(z) = \log|z|$.