Convergence analysis of some staggered schemes for the compressible stationary barotropic Navier-Stokes equations

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with

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Aim

Stationary compressible Navier-Stokes equations in a bounded open set of $\mathbb{R}^d$, $d = 2$ or 3, with EOS $p = \rho^\gamma$.

- $\gamma > 1$ if $d = 2$.
- $\gamma > 3$ if $d = 3$ (aim: $\gamma > \frac{3}{2}$).

Discretization by a numerical scheme

- MAC scheme : FV on staggered grid, often used in CFD codes: simple, cheap, and robust.
- Nonconforming FE (Crouzeix Raviart) on velocity and FV on density, used at IRSN.

$\rightsquigarrow$ Existence of solutions (P. L. Lions, E. Feireisl, A. Novotny & I. Strabaska ...)
No uniqueness result.

Stationary compressible Navier Stokes equations

Ω : bounded open set of $\mathbb{R}^d$, $d = 2$ or 3,
with Lipschitz continuous boundary $\partial\Omega$,
$\gamma \geq 1$, $\mathbf{f} \in L^2(\Omega)^d$ and $M > 0$

$$\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \Delta\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega, \quad \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega, \tag{MOM}$$

$$\mathrm{div}(\rho\,\mathbf{u}) = 0 \text{ in } \Omega, \; \rho \geq 0 \text{ in } \Omega, \; \int_\Omega \rho(x) = M, \tag{MASS}$$

$$p = \rho^\gamma \text{ in } \Omega \tag{EOS}$$

Functional spaces if $d = 2$, of $d = 3, \gamma \geq 3$:
$u \in H_0^1(\Omega)^d$, $p \in L^2(\Omega)$, $\rho \in L^{2\gamma}(\Omega)$.

If $d = 3$ and $\frac{3}{2} < \gamma < 3$: $p \in L^{\frac{3(\gamma-1)}{\gamma}}(\Omega)$, $\rho \in L^{3(\gamma-1)}(\Omega)$.
If $d = 3$ and $\gamma = \frac{3}{2}$, $p \in L^1(\Omega)$, $\rho \in L^\gamma(\Omega)$.

Weak formulation

Functional spaces : $\mathbf{u} \in H_0^1(\Omega)^d$, $p \in L^2(\Omega)$, $\rho \in L^{2\gamma}(\Omega)$

- Momentum equation:

For all $\mathbf{v} \in H_0^1(\Omega)^d$,

$$\int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} - \int_\Omega \rho \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v} \, dx - \int_\Omega p \, \mathrm{div} \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \quad (\mathrm{MOM})_w$$

- Mass equation:

$$\int_\Omega \rho \, \mathbf{u} \cdot \nabla \varphi = 0 \text{ for all } \varphi \in C_c^\infty(\Omega) \qquad (\mathrm{MASS})_w$$

$$\rho \geq 0 \text{ a.e.}, \quad \int_\Omega \rho = M$$

- EOS: $p = \rho^\gamma$

Main result

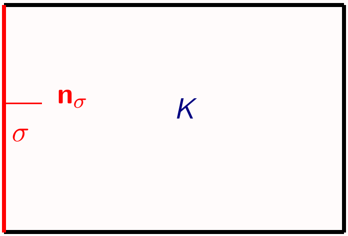
- Discretization of (MOM) : grid cell pressure and density unknowns: $(p_K)_{K\in\mathcal{T}}$, $(\rho_K)_{K\in\mathcal{T}}$
  $\rightsquigarrow$ MAC scheme (most commonly used scheme for incompressible and compressible Navier Stokes equations) velocity unknowns $(u_\sigma)_{\sigma\in\mathcal{E}}$ approximation of the normal velocities
  $\rightsquigarrow$ Crouzeix-Raviart Finite Element (used by IRSN) velocity unknowns $(\mathbf{u}_\sigma)_{\sigma\in\mathcal{E}}$ approximation of the full velocities
- Discretization of the mass equation (and total mass constraint) by classical upwind Finite Volume
- Existence of a solution for the discrete problem.
- Proof of the convergence (up to subsequence) of the solution of the discrete problem towards a weak solution of the continuous problem (no uniqueness result for this problem) as the mesh size goes to 0

MAC scheme, choice of the discrete unknowns

- $\mathcal{T}$: Cartesian rectangular mesh of Ω, mesh size: h
  $\mathcal{E}$: edges of $\mathcal{T}$
- Discretization of $\mathbf{u}$, p and ρ by piecewise constant functions.

$\mathbf{n}_\sigma$ is the normal vector to σ, with $\mathbf{n}_\sigma \geq \mathbf{0}$.

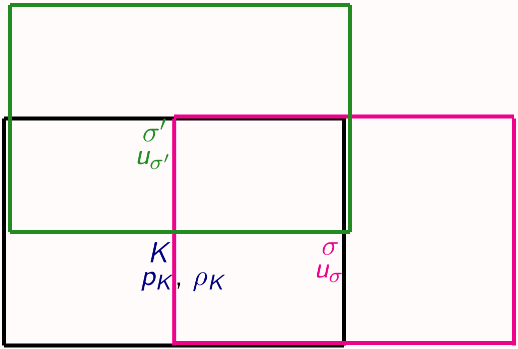
Unknowns for $\mathbf{u}_\mathcal{T}$:
u_σ, $\sigma \in \mathcal{E}$. u_σ is an approximate value for $\mathbf{u} \cdot \mathbf{n}_\sigma$ ($u_\sigma \in \mathbb{R}$)
$u_\sigma = 0$ if $\sigma \subset \partial\Omega$



Unknowns for $p_\mathcal{T}$ and $\rho_\mathcal{T}$:
p_K, ρ_K, $K \in \mathcal{T}$

MAC scheme, discrete functional spaces, $d = 2$

- $p_{\mathcal{T}}, \rho_{\mathcal{T}} \in X_{\mathcal{T}}$, $p_{\mathcal{T}} = p_K$, $\rho_{\mathcal{T}} = \rho_K$ in K, $K \in \mathcal{T}$ (black cell)
- $\mathbf{u}_{\mathcal{T}} = (u^{(1)}_{\mathcal{T}}, u^{(2)}_{\mathcal{T}}) \in \mathbf{H}_{\mathcal{T}}$
  $u^{(1)}_{\mathcal{T}} = u_\sigma$ in the pink cell
  $u^{(2)}_{\mathcal{T}} = u_\sigma$ in the green cell



Discretization of the momentum equation: diffusive term

$\mathbf{u}, \mathbf{v} \in \mathbf{H}_{\mathcal{T}}$, the discretization of $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{T}}$ is:

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{T}} = \langle u_1, v_1 \rangle_{\mathcal{T}_1} + \langle u_2, v_2 \rangle_{\mathcal{T}_2}$$

where $\langle u_1, v_1 \rangle_{\mathcal{T}_1}$ is the usual inner product associated to the discrete FV Laplace operator on the mesh $\mathcal{T}_1$, that is:

$$\langle u_1, v_1 \rangle_{\mathcal{T}_1} = -\int_{\Omega} \Delta_{\mathcal{T}_1} u_1 v_1 = \sum_{\sigma_1 = (K_1, L_1)} |\sigma_1| d_{\sigma_1} \frac{u_{K_1} - u_{L_1}}{d_{\sigma_1}} \frac{v_{K_1} - v_{L_1}}{d_{\sigma_1}}$$

Discretization of the momentum equation : gradient term

- $\mathbf{v} \in H_{\mathcal{T}}$. $\mathrm{div}_{\mathcal{T}}\mathbf{v}$ is constant on K, $K \in \mathcal{T}$ and

$$|K|\mathrm{div}_K\mathbf{v} = \sum_{\sigma\in\mathcal{E}_K} \mathrm{sg}_{K,\sigma} v_\sigma |\sigma|$$

  $\mathrm{sg}_{K,\sigma} = \mathrm{sign}(\mathbf{n}_\sigma \cdot \mathbf{n}_{K,\sigma})$, $\mathbf{n}_{K,\sigma}$ is the normal vector to σ, outward to K

- Pressure gradient term $\displaystyle\int_\Omega p \mathrm{div}\mathbf{v} dx$ is then discretized by

  $\displaystyle\int_\Omega p_{\mathcal{T}} \mathrm{div}_{\mathcal{T}} \mathbf{v}_{\mathcal{T}} dx = \sum_{K\in\mathcal{T}} |K| p_K \mathrm{div}_K \mathbf{v}$

Discretization of the mass equation $\operatorname{div}(\rho \mathbf{u}) = 0$, $\int_\Omega \rho = M$

$$\text{For } K \in \mathcal{T},\ \int_{\partial K} \rho \mathbf{u} \cdot \mathbf{n}_{K,\sigma} = 0, \qquad \sum_{K \in \mathcal{T}} \int_\Omega \rho = M$$

$$\Downarrow$$

$$\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} + M_K = 0$$

- discrete mass flux $F_{K,\sigma} = |\sigma| \rho_\sigma \mathrm{sg}_{K,\sigma} u_\sigma$
- and upwind choice for ρ_σ, that is

$$\rho_\sigma = \begin{cases} \rho_K \text{ if } u_\sigma \geq 0, \\ \rho_L \text{ if } u_\sigma < 0, \end{cases} \qquad \sigma = K|L$$

- $M_K = |K| h^\alpha (\rho_K - \frac{M}{|\Omega|})$ with $\rho_K^* = \frac{M}{|\Omega|}$ and $\alpha > 0$.

$$|K| h^\alpha (\rho_K - \rho_K^*) + \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} = 0 \qquad (\mathrm{MASS})_\mathcal{T}$$

Upwinding is enough to ensure (with M) existence (and uniqueness) of a positive solution $\rho_\mathcal{T}$, to the discrete mass equation, for a given $\mathbf{u}_\mathcal{T}$.

Discretization of the momentum equation : nonlinear convection term

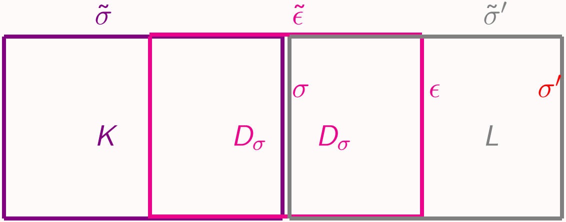
Nonlinear convection operator compatible with mass balance $\rightsquigarrow$ kinetic energy control in the transient case.
Discrete equivalent of the continuous (formal) result:

$$\left.\begin{array}{l}\partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \mathbf{u} = 0 \text{ on } \partial\Omega\end{array}\right\} \Rightarrow \int_\Omega (\partial_t(\rho z) + \operatorname{div}(\rho \mathbf{u} z)) = \frac{1}{2}\frac{d}{dt}\int_\Omega \rho z^2.$$

Problem: mass balance discretized on p, ρ cells,
nonlinear convection operator on u_1, u_2 velocity cells.

Solution: Ad hoc computation of the fluxes on the u_1, u_2 cell boundaries

Example: nonlinear convection term on a u_1 cell (D_σ):



Half sum of mass balances on K and L yields mass balance on D_σ:

$$h^\alpha |K_\sigma| (\tilde{\rho}_\sigma - \tilde{\rho}^*_\sigma) + \sum F_{\sigma,\epsilon} = 0$$

$\tilde{\rho}_\sigma = \frac{1}{2|K_\sigma|} \left(|K|\rho_K + |L|\rho_L \right)$, $\tilde{\rho}^*_\sigma = \frac{M}{|\Omega|}$, $F_{\sigma,\epsilon} = \frac{1}{2} \left(F_{K,\sigma} + F_{K,\sigma'} \right)$

Compatible discretization of momentum balance:

$$C^{(1)}_\sigma = h^\alpha |K_\sigma| (\tilde{\rho}_\sigma - \tilde{\rho}^*_\sigma) u_\sigma + \sum_{\epsilon \subset \partial K_\sigma} F_{\sigma,\epsilon} u_\sigma$$

$$\mathbf{C}_{\mathcal{T}} = \left(C^{(1)}_{\mathcal{T}}, C^{(2)}_{\mathcal{T}} \right)^t$$

Discretization of the EOS

Upwinding in the mass equation $\rightsquigarrow \rho_K > 0$ for all K

Discretization of the EOS:

$$p_K = \rho_K^\gamma \text{ for all } K \in \mathcal{T}$$

The MAC scheme for the Navier-Stokes equations

$$\left\{\begin{aligned}
&\mathbf{u}_\mathcal{T} \in \mathbf{H}_\mathcal{T}, p_\tau \in X_\mathcal{T}, \rho_\tau \in X_\mathcal{T}\\
&\int_\Omega \mathbf{C}_\mathcal{T}\mathbf{v} + \langle \mathbf{u}, \mathbf{v}\rangle_\mathcal{T} - \int_\Omega p_\mathcal{T} \mathrm{div}_\mathcal{T}\mathbf{v} = \int_\Omega \mathbf{f}\mathbf{v}, \forall \mathbf{v} \in \mathbf{H}_\mathcal{T}\\
&|K| h^\alpha(\rho_K - \rho_K^*) + \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} = 0\\
&p_K = \rho_K^\gamma \text{ for all } K \in \mathcal{T}
\end{aligned}\right. \qquad (\mathrm{NS}_\mathcal{T})$$

Convergence result: $d = 2, \gamma > 1$ or $d = 3, \gamma > 3$, then as $h_\mathcal{T} \to 0$, up to a subsequence:

- $\mathbf{u}_\mathcal{T} \to \mathbf{u}$ in $L^2(\Omega)^d$, $u \in H_0^1(\Omega)^d$
- $p_\mathcal{T} \to p$ in $L^q(\Omega)$ for any $1 \leq q < 2$ and weakly in $L^2(\Omega)$
- $\rho_\mathcal{T} \to \rho$ in $L^q(\Omega)$ for any $1 \leq q < 2\gamma$ and weakly in $L^{2\gamma}(\Omega)$

where $(\mathbf{u}, p, \rho)$ is a weak solution of the compressible Navier-Stokes equations

Proof of convergence, main steps

1. Discrete $H^1_0(\Omega)$ estimate on the components of $\mathbf{u}_{\mathcal{T}}$
2. $L^2(\Omega)$ estimate on $p_{\mathcal{T}}$ and $L^{2\gamma}(\Omega)$ estimate on $\rho_{\mathcal{T}}$

   These two steps give (up to a subsequence), as $h \to 0$,

   - $\mathbf{u}_{\mathcal{T}} \to \mathbf{u}$ in $L^2(\Omega)$ and $\mathbf{u} \in H^1_0(\Omega)^d$
   - $p_{\mathcal{T}} \to p$ weakly in $L^2(\Omega)$
   - $\rho_{\mathcal{T}} \to \rho$ weakly in $L^{2\gamma}(\Omega)$
3. $(\mathbf{u}, p, \rho)$ is a weak solution of $\mathrm{div}(\rho\mathbf{u} \otimes \mathbf{u}) - \Delta\mathbf{u} + \nabla p = \mathbf{f}$,
   $\mathrm{div}(\rho\mathbf{u}) = 0$
   $\rho \geq 0$, $\int_\Omega \rho = M$
4. Main difficulty: passage to the limit in $p_{\mathcal{T}} = \rho_{\mathcal{T}}^{\gamma}$; proven by "strong" convergence of $p_{\mathcal{T}}$ and $\rho_{\mathcal{T}}$

Hints for the proof of convergence

Assume estimates on u_n, p_n, ρ_n, and that as $n \to \infty$:

$$u_n \to u \text{ in } L^2(\Omega)^d \text{ and weakly in } H_0^1(\Omega)^d,$$

$$p_n \to p \text{ weakly in } L^2(\Omega),$$

$$\rho_n \to \rho \text{ weakly in } L^{2\gamma}(\Omega).$$

Assume that u, p and ρ satisfy (MASS) and (MOM).

$$p_n = \rho_n^\gamma \qquad \text{(EOS)}_{\mathcal{T}}$$

How to pass to the limit to obtain (EOS) $p = \rho^\gamma$?
Key ideas, assuming u_n, p_n, ρ_n to be regular solutions of

$$\mathrm{div}(\rho_n \mathbf{u}_n \otimes \mathbf{u}_n) - \Delta \mathbf{u}_n + \mathbf{u}_n \nabla p_n = \mathbf{f}_n \text{ in } \Omega, \;\; \mathbf{u}_n = \mathbf{0} \text{ on } \partial\Omega, \qquad \text{(MOM)}_n$$

$$\mathrm{div}(\rho_n \, \mathbf{u}_n) = 0 \;\text{ in } \Omega, \; \rho \geq 0 \;\text{ in } \Omega, \; \int_\Omega \rho_n(x) = M_n, \qquad \text{(MASS)}_n$$

$$p_n = \rho_n^\gamma \text{ in } \Omega \qquad \text{(EOS)}_n$$

with $\mathbf{f}_n \to \mathbf{f}$ in $L^2(\Omega)$ and $M_n \to M$.

Proof of EOS, idea # 1

$p_n = \rho_n^\gamma \text{ in } \Omega$

p_n and ρ_n converge only weakly... and $\gamma > 1$

Idea # 1 : (for $d = 2$ or $d = 3$, $\gamma \geq 3$) prove $\int_\Omega p_n \rho_n \to \int_\Omega p\rho$ and deduce a.e. convergence (of p_n and ρ_n) and $p = \rho^\gamma$.

(For $d = 3$, $\gamma \leq 3$, use $p_n \rho_n^\theta$).

In the sequel, we take $d = 3$, $\gamma > 3$.
Assume we have shown that $\int_\Omega p_n \rho_n \to \int_\Omega p\rho$.
Then using the fact that $y \mapsto y^\gamma$ is an increasing function on $\mathbb{R}_+$,

$\rho_n \to \rho \text{ in } L^q(\Omega) \text{ for all } 1 \leq q < 2\gamma$,

$p_n = \rho_n^\gamma \to \rho^\gamma \text{ in } L^q(\Omega) \text{ for all } 1 \leq q < 2$,

and $p = \rho^\gamma$.

Proof of EOS, idea # 2 : use momentum equation

Take test function $\mathbf{v}_n \in H_0^1(\Omega)^d$ such that $\mathrm{div}\mathbf{v}_n = \rho_n$
Use $\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} = \int_\Omega \mathrm{div}\mathbf{u}\ \mathrm{div}\mathbf{v} + \int_\Omega \mathrm{curl}\mathbf{u} \cdot \mathrm{curl}\mathbf{v},\ \forall \mathbf{u}, \mathbf{v} \in H_0^1(\Omega)^d$

$$\int_\Omega \mathrm{div}\mathbf{u}_n \mathrm{div}\mathbf{v}_n + \int_\Omega \mathrm{curl}\mathbf{u}_n \cdot \mathrm{curl}\mathbf{v}_n - \int_\Omega p_n \mathrm{div}\mathbf{v}_n$$
$$= \int_\Omega \rho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{v}_n + \int_\Omega \mathbf{f}_n \cdot \mathbf{v}_n.$$

Assume we can also choose $\mathbf{v}_n$ such that $\mathrm{curl}\mathbf{v}_n = 0$ and $(\mathbf{v}_n)_{n\in\mathbb{N}}$ bounded in $H_0^1(\Omega)^d$, then, up to a subsequence,
$\mathbf{v}_n \to v$ in $L^2(\Omega)^d$ and weakly in $H_0^1(\Omega)^d$,
$\mathrm{curl}(\mathbf{v}) = 0$, $\mathrm{div}(\mathbf{v}) = \rho$.

$$\int_\Omega (\mathrm{div}(\mathbf{u}_n) - p_n)\rho_n = \int_\Omega \rho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{v}_n + \int_\Omega \mathbf{f}_n \cdot \mathbf{v}_n.$$

If we prove that $\int_\Omega \rho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{v}_n \to \int_\Omega \rho \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v}$ then:

$$\lim_{n\to\infty} \int_\Omega (\mathrm{div}\mathbf{u}_n - p_n)\, \rho_n = \int_\Omega \rho \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v} + \int_\Omega \mathbf{f} \cdot \mathbf{v}.$$

$$\lim_{n\to\infty}\int_\Omega (\mathrm{div}\mathbf{u}_n - p_n)\rho_n = \int_\Omega \rho\mathbf{u}\otimes\mathbf{u} : \nabla\mathbf{v} + \int_\Omega \mathbf{f}\cdot\mathbf{v}$$
$$= \int_\Omega \mathrm{div}\mathbf{u}\ \mathrm{div}\mathbf{v} + \int_\Omega \mathrm{curl}\mathbf{u}\cdot\mathrm{curl}\mathbf{v} - \int_\Omega p\mathrm{div}\mathbf{v}$$
$$= \int_\Omega (\mathrm{div}\mathbf{u} - p)\rho$$

Lemma $\beta > 1$ $\rho \in L^{2\beta}(\Omega)$, $\rho \geq 0$ a.e. in Ω, $u \in (H^1_0(\Omega))^d$, $\mathrm{div}(\rho\mathbf{u}) = 0$, then:

$$\int_\Omega \rho\mathrm{div}\mathbf{u}dx = 0$$
$$\int_\Omega \rho^\beta\mathrm{div}\mathbf{u}dx = 0$$

$$\Downarrow$$

$$\lim_{n\to\infty}\int_\Omega p_n\ \rho_n = \int_\Omega p\ \rho$$

Proof of EOS, $\int_\Omega \rho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{v}_n \to \int_\Omega \rho \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v}$

Since $\mathrm{div}(\rho_n \mathbf{u}_n) = 0$

$$\int_\Omega \rho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{v}_n = \int_\Omega (\rho_n \mathbf{u}_n \cdot \nabla) \mathbf{u}_n \cdot \mathbf{v}_n,$$

and the sequence $((\rho_n \mathbf{u}_n \cdot \nabla)\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $L^r(\Omega)^d$ with $\frac{1}{r} = \frac{1}{2\gamma} + \frac{1}{6} + \frac{1}{2}$ (and $r > \frac{6}{5}$ since $\gamma > 3$).
Then, up to a subsequence $(\rho_n \mathbf{u}_n \cdot \nabla)\mathbf{u}_n \to \Psi$ weakly in $L^r(\Omega)^d$.
and $v_n \to v$ in $L^q(\Omega)^d$ for all $q < 6$,

$$\int_\Omega (\rho_n \mathbf{u}_n \cdot \nabla) \mathbf{u}_n \cdot \mathbf{v}_n \to \int_\Omega \Psi \cdot \mathbf{v}$$

But, $\forall \mathbf{w} \in H_0^1(\Omega)^d$,

$$\int_\Omega (\rho_n \mathbf{u}_n \cdot \nabla) \mathbf{u}_n \cdot \mathbf{w} = \int_\Omega \rho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{w} \to \int_\Omega \rho \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{w}.$$

$$\Downarrow$$

$$\Psi = (\rho \mathbf{u} \cdot \nabla)\mathbf{u}.$$

Generalizations

- (Easy) Complete Diffusion term: $-\mu\Delta\mathbf{u} - \frac{\mu}{3}\nabla(\mathrm{div}\,\mathbf{u})$, with $\mu \in \mathbb{R}_+^\star$ given, instead of $-\Delta\mathbf{u}$.
- (Ongoing work) Navier-Stokes Equations with $d = 3$ and $\frac{3}{2}\gamma < \gamma \leq 3$. (probably sharp result with respect to γ without changing the diffusion term or the EOS)
- (Ongoing work) More general EOS
- (Open question) Other boundary conditions. Addition of an energy equation
- (Open question) Evolution equation (Stokes and Navier-Stokes)

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Preliminary lemma with the numerical scheme (1)

Roughly speaking, upwinding replaces $\mathrm{div}(\rho \mathbf{u}) = 0$ and $\int_\Omega \rho dx = M$ by

$$\mathrm{div}(\rho \mathbf{u}) - h\mathrm{div}(|\mathbf{u}|\nabla\rho) + h^\alpha(\rho - \rho^\star) = 0$$

with $\rho^\star = \frac{M}{|\Omega|}$

This equation has (for a given u) a solution $\rho > 0$ and we prove

$$\int_\Omega \rho_n^\gamma \mathrm{div}_n \mathbf{u}_n dx \leq Ch^\alpha,$$

$$\int_\Omega \rho_n \mathrm{div}_{\mathcal{T}} \mathbf{u}_n dx \leq Ch^\alpha.$$

C depends on Ω, M and γ

Ch^α is due to $h^\alpha(\rho - \rho^\star)$

$\leq$ due to upwinding

The first inequality leads to the estimate on the approx. solution.

Preliminary lemma with the numerical scheme (2)

For the passage to the limit on the EOS

$$\int_\Omega \rho_n \mathrm{div}_\mathcal{T}\, \mathbf{u}_\mathcal{T} dx \le Ch^\alpha$$

$$\int_\Omega \rho \mathrm{div}\, \mathbf{u} dx = 0$$

give $\lim_{n\to\infty} \int_\Omega p_n \rho_n dx \le \int_\Omega p\rho dx = 0$,
which is sufficient to prove the a.e. convergence (up to a subsequence) of p_n and ρ_n

Passage to the limit in the EOS with the Mac scheme

Miracle with the Mac scheme:

There exists a discrete counterpart of
$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} dx = \int_\Omega (\mathrm{div} \mathbf{u} \mathrm{div} \mathbf{v} + \mathrm{curl} \mathbf{u} \cdot \mathrm{curl} \mathbf{v}) dx$

Passage to the limit in the EOS with Crouzeix-Raviart

No discrete counterpart with Crouzeix-Raviart. Two possible solutions

- Use the continuous equality. This is possible with an additional regularization term in the mass equation (not needed from the numerical point of view, only needed to prove the convergence), less diffusive than the upwinding.
- Discretize $\int_\Omega (\mathrm{div}\mathbf{u}\mathrm{div}\mathbf{v} + \mathrm{curl}\mathbf{u} \cdot \mathrm{curl}\mathbf{v}) dx$ instead of $\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} dx$. Better for passing to the limit in the EOS but the discretized momentum equation is not coercive (with Crouzeix-Raviart Finite Element). One needs a penalization term in the discrete momentum equation (crucial from the numerical point of view). cf. Karlsen-Karper work for the compressible Stokes problem.