W-Graphs, Nilpotent Orbits, and Primitive Ideals

Birne Binegar
Oklahoma State University
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Abstract: Let G be the real points of a connected linear reductive complex algebraic group defined over $\mathbb{R}$ and let $\widehat{G}_{adm,\lambda}$ be the set of equivalences classes of irreducible admissible representations of G of infinitesimal character λ, which we assume to be regular and integral. The Atlas software enumerates the representations in $\widehat{G}_{adm,\lambda}$, and computes the Kazhdan-Lusztig-Vogan polynomials $P_{x,y}(q)$ which not only prescribe the Jordan-Hölder decomposition of standard modules in terms of the irreducibles in $\widehat{G}_{\lambda,adm}$, but can also be used to endow the set $\widehat{G}_{adm,\lambda}$ with the structure of a W-graph, a certain weighted directed graph. The strongly connected components of this W-graph are W-cells. In this talk I will describe how the W-graph structure of an W-cell $\mathcal{C}$ allows one to compute the (common) associated variety of the annihilators of the representations in $\mathcal{C}$, and in fact, to determine when two representations $x, y \in \mathcal{C}$ share the same annihilator.

1. Setup

Let G be the real points of a connected linear reductive complex algebraic group $\mathbb{G}$ defined over $\mathbb{R}$. We fix a Cartan involution θ of G and let $K = G^{\theta}$ be the corresponding maximal compact subgroup. θ extends to holomorphic involution of $\mathbb{G}$ and we set $\mathbb{K} = \mathbb{G}^{\theta}$. Let $\mathfrak{g}$ denote the complexified Lie algebra of G and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote by W the Weyl group of $\mathfrak{g}$.

Let $\widehat{G}_{adm}$ denote the set of equivalence classes of irreducible admissible representations of G and let $\mathcal{HC}$ denote the corresponding set of irreducible Harish-Chandra modules - obtained by sending an irreducible admissible representation $x \in \widehat{G}_{adm}$ to its space V_x of smooth K-finite vectors and then regarding the latter as an irreducible $(\mathfrak{g}, K)$-module.

In this talk we shall be concerned with explicitly attaching certain algebraic invariants to the representations in $\widehat{G}_{adm}$.

The most basic of these algebraic invariants is the infinitesimal character of a representation. Via the Harish-Chandra homomorphism ϕ_{HC} the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ can be identified, as a commutative algebra, with the W-invariant polynomials on $\mathfrak{h}^*$. On the other hand, by Schur's lemma, any $z \in Z(\mathfrak{g})$ acts by a certan scalar $c_{V_x,z}$ on an irreducible $(\mathfrak{g}, K)$-module V_x. Thus, to any particular irreducible $(\mathfrak{g}, K)$-module V_x we have a homomorphism $S(\mathfrak{h})^W \to \mathbb{C} : z \mapsto c_{V_x,z}$ which in turn can be identified with the evaluation of $\phi_{HC}(z)$ at a point of a particular W-orbit in $\mathfrak{h}^*$. In this way we have a map from $infchar : \widehat{G}_{adm} \to \mathfrak{h}^*/W$. For $\lambda \in \mathfrak{h}^*/W$ we denote by

$$\widehat{G}_{adm,\lambda} = \left\{ x \in \widehat{G}_{adm} \mid infchar(x) = \lambda \right\}$$

and refer to the representations in $\widehat{G}_{adm,\lambda}$ as the *irreducible admissible representations of infinitesimal character* λ. Evidently,

$$\widehat{G}_{adm} = \coprod_{\lambda \in \mathfrak{h}^*/W} \widehat{G}_{adm,\lambda}$$

$\widehat{G}_{adm,\lambda}$ being the set of irreducible admissible representations with infinitesimal character λ. By a theorem of Harish-Chandra, each $\widehat{G}_{adm,\lambda}$ is a finite set.

When λ is regular and integral (meaning $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}_{>0}$ for any simple coroot $\check{\alpha}$), the Atlas software can enumerate the irreducible representations in $\widehat{G}_{adm,\lambda}$. In fact, the Atlas software endows the set $\widehat{G}_{adm,\lambda}$

(λ again regular and integral) with a certain graph structure. The main point of this talk is to show how this graph structure can in turn be used to explicitly attach nilpotent orbits and primitive ideals to the admissible representations in $\widehat{G}_{adm,\lambda}$. I should perhaps point out that it what follows it is most helpful to think of the representations in $\widehat{G}_{adm,\lambda}$ (as well as the corresponding Harish-Chandra modules $\mathcal{HC}$) as having been explicitly enumerated $0,1,2,\ldots,n$ (which is in fact exactly how the Atlas software enumurates $\widehat{G}_{adm,\lambda}$.

2. Blocks and Cells

Henceforth, we fix $\lambda \in \mathfrak{h}^*$ to be a regular and integral. As we have remarked above, the Atlas software can enumerate the irreducible admissible representations with infinitesimal character λ. In fact, the software does a lot more than simply count representations, it also endows the set $\widehat{G}_{adm,\lambda}$ with a certain directed, weighted, graph structure.

The way in which Atlas actually enumerates the representations in $\widehat{G}_{adm,\lambda}$ is described in detail in [Adams-du Cloux]. For our purposes here, we do not need to understand the connection between Atlas's enumeration and more conventional Langland's parameters; however, it is helpful to have a grasp of the basic combinatorial data that goes into the Atlas enumeration. As described in Adams-du Cloux, an irreducible admissible representation is corresponds to a certain pair (x,y) where x is a $\mathbb{K}$-orbit in $\mathbb{G}/\mathbb{B}$ and y is a $\mathbb{K}^\vee$-orbit in $\mathbb{G}^\vee/\mathbb{B}^\vee$; here $\mathbb{B}$ is a Borel subgroup of $\mathbb{G}$, $\mathbb{G}^\vee$ is the dual group of $\mathbb{G}$, $\mathbb{B}^\vee$ a Borel subgroup of $\mathbb{G}^\vee$, and $\mathbb{K}^\vee$ is the complexification of a real form of a maximal compact subgroup $K^\vee$ of a real form $G^\vee$ of $\mathbb{G}^\vee$.

The Atlas software enumerates the admissible representations in $G_{adm,\lambda}$ by first enumerating the $\mathbb{K}$-orbits in $\mathbb{G}/\mathbb{B}$ and then for each real form $G^\vee$ of the dual group $\mathbb{G}^\vee$ enumerating the $\mathbb{K}^\vee$–orbits in $\mathbb{G}^\vee/\mathbb{B}^\vee$ and checking a certain compatibility conditions. The collection of irreducible admissible representations arise in this fashion from a particular real form on the dual side constitute what is called a **block** of representations, and one has

$$\widehat{G}_{adm,\lambda} \approx \coprod_{\text{dual real forms } G^\vee} \{(x,y) \mid x \in \mathbb{K}\backslash\mathbb{G}/\mathbb{B} \quad , \quad y \in \mathbb{K}^\vee\backslash\mathbb{G}^\vee/\mathbb{B}^\vee \quad , \quad x,y \text{ compatible}\}$$

We remark that in a similar fashion, the irreducible admissible representations of a real form $G^\vee$ of $\mathbb{G}^\vee$, can be broken up into blocks corresponding to various real forms of $\mathbb{G}$. In fact, Vogan duality tells us that if (x,y), $x \in \mathbb{K}\backslash\mathbb{G}/\mathbb{B}$, $y \in \mathbb{K}^\vee\backslash\mathbb{G}^\vee/\mathbb{B}^\vee$ corresponds to an admissible representation of G, then (y,x) corresponds to an admissible representation of the real form $G^\vee$ of $\mathbb{G}^\vee$ corresponding to $\mathbb{K}^\vee$. At any rate, in the Atlas world, irreducible admissible representations come in blocks corresponding to pairs $(G,G^\vee)$.

Example 2.1. Let $\mathbb{G}$ be the simply connected complex linear group with Lie algebra E_8. We note that $\mathbb{G}$ is self-dual and has three real forms: the compact real form e_8, the quaterionic real form $E_8\left(\mathfrak{e}_7 + \mathfrak{su}\left(2\right)\right)$ and the split real form $E_8\left(\mathbb{R}\right)$. Below is a table showing the number of irreducible admissible representations (at regular infinitesimal character λ) in each block of $\mathbb{G}$

	$\mathfrak{e}_8$	$E_8\left(e_7+\mathfrak{su}\left(2\right)\right)$	$E_8\left(\mathbb{R}\right)$
$\mathfrak{e}_8$	0	0	1
$E_8\left(e_7+\mathfrak{su}\left(2\right)\right)$	0	3150	73410
$E_8\left(\mathbb{R}\right)$	1	73410	453060

The total number of irreducible admissible representations of $E_8\left(\mathbb{R}\right)$ with infinitesimal character λ is thus

$$1 + 73410 + 453060 = 526471 \quad .$$

Similarly, at any fixed regular integral infinitesimal character λ, there are $3250 + 73410 = 76\,660$ irreducible admissible representations of the quaterionic real form.

Each block of representation can in turn be partitioned into smaller subsets called cells. We describe in a minute how Atlas does this; but we'll start with a purely formal definition.

Definition 2.2. *Given two irreducible Harish-Chandra modules x, y in $\mathcal{HC}_\lambda$, we shall say that $x \rightsquigarrow y$ if there exists an irreducible finite-dimensional representation F of G such that F occurs in the tensor algebra of $\mathfrak{g}$ and x occurs as a subquotient of $y \otimes F$. We say that*

$$x \sim y$$

*if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The equivalence classes for the relation $\sim$ are called **cells** of Harish-Chandra modules, or W-cells.*

Alternatively, one can define W-cells by putting a certain graph structure on the set $\mathcal{HC}_\lambda$.

Definition 2.3. *Given $x, y \in \mathcal{HC}_\lambda$, we say that there is a (directed) edge $x \longrightarrow y$ from x to y whenever x occurs as a subquotient of $y \otimes \mathfrak{g}$.*

It is easy to see that the transitive closure of "$\longrightarrow$" replicates the relation "$\rightsquigarrow$" of Definition (??) and that the strongly connected components (that is the bidirectionally connected) of the graph with vertices $x \in \mathcal{HC}_\lambda$ and edges $x \longrightarrow y$ as defined above coincide with the W-cells of $\mathcal{HC}_\lambda$. It also turns out that each connected component of this graph coincides with a particular block of representations. The Atlas software computes this graphical structure, block by block (connected component by connected component) essentially as a by-product its computation of the Kazhdan-Lusztig-Vogan (KLV-) polynomials for the various real forms of $\mathbb{G}$.

In fact, the Atlas software endows the set $\mathcal{HC}_\lambda$ with an even more elaborate graphical structure.

Definition 2.4. *Let B be a block of irreducible Harish-Chandra modules of inf char λ.*

*The W-**graph** of B is the weighted digraph where:*

- *the **vertices** are the elements $x \in B$*
- *there is an **edge** $x \to y$ of **multiplicity** m between two vertices if*

$$\text{coefficient of } q^{(|x|-|y|-1)/2} \text{ in } P_{x,y}(q) = m \neq 0$$

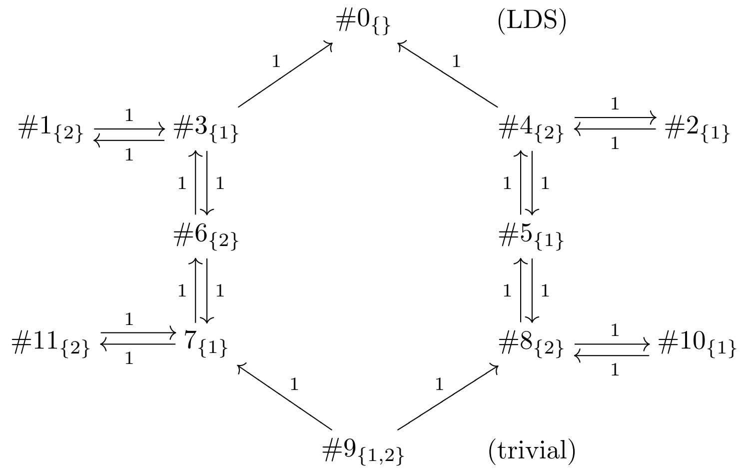
- *there is assigned to each vertex x a subset $\tau(x)$ of the set of simple roots of $\mathfrak{g}$, the **descent set** of x.*

Remark 2.5. This W-graph structure, although computed via KLV polynomials, is in fact a significant enhancement of the graph structure we defined before in purely representation theoreticl terms. In particular,

- The multiplicity of an edge $x \to y$ coincides with the multiplicity with which the representation x occurs in $y \otimes \mathfrak{g}$.
- The descent set of a vertex x coincides with the tau invariant of the annihilator of x in $U(\mathfrak{g})$.

Example 2.6. Let $G_2(\mathbb{R})$ be the split real form of the simply connected complex group with Lie algebra G_2. G_2 is self-dual, and the output of Atlas's `wgraph` command for the big block $G_2(\mathbb{R}) \times G_2(\mathbb{R})$ is block descent edge vertices, element set multiplicities 0 1 2 (3,1) 2 1 (4,1) 3 1 (0,1),(1,1),(6,1) 4 2 (0,1),(2,1),(5,1) 5 1 (4,1),(8,1) 6 2 (3,1),(7,1) 7 1 (6,1),(11,1) 8 2 (5,1),(10,1) 9 1,2 (7,1),(8,1) 10 1 (8,1) 11 2 (7,1)

This data leads to the following W-graph



Note that there are four strongly connected components to this graph

$$\{\#0\} \quad , \quad \{\#1, \#3, \#6, \#7, \#11\} \quad , \quad \{\#2, \#4, \#5, \#8, \#10\} \quad , \quad \{\#9\}$$

and thus four W-cells in this block. I might also point out, to help you get your bearings, that, at infinitesimal character ρ, vertex #0 corresponds to the large discrete series representation while vertex #0 corresponds to the trivial representation.

3. From W-graphs to invariants

I'll now describe how one can utilize the W-graph structure on $\widehat{G}_{adm,\lambda}$ to determine certain algebraic invariants for the representations in $\widehat{G}_{adm,\lambda}$. Before beginning though I would like to point out what to me seems most remarkable: the algebraic invariants attached to a vertex are not determined by properties of the vertex *per se* but rather by how the vertex is situated in W-graph of the block that contains it.

3.1. **Primitive Ideals.** The invariants to be considered here will actually be invariants attached most directly to the primitive ideal in $U(\mathfrak{g})$. I point out, however, that there are other invariants (e.g. the associated variety of an irreducible Harish-Chandra module [21]) that are attached directly to Harish-Chandra modules themselves and which are in fact an invariant of an entire cell of Harish-Chandra modules and so are at least partially determined by the W-graph structure of the block in which a representation sits.

Definition 3.1. *Let V be an irreducible $U(\mathfrak{g})$-module.*

$$Ann(V) := \{X \in U(\mathfrak{g}) \mid Xv = 0 \quad , \quad \forall\ v \in V\}$$

*is a two-sided ideal in $U(\mathfrak{g})$. It is called the **primitive ideal** in $U(\mathfrak{g})$ attached to V.*

It is easy to see that $Ann(V) = Ann(V') \Longrightarrow inf\ ch\ V = inf\ ch\ V'$ and so it makes sense to talk about the primitive ideals at infinitesimal character λ. We set

$$Prim(\mathfrak{g})_\lambda := \text{ set of primitive ideals in } U(\mathfrak{g}) \text{ with infinitesimal character } \lambda$$

For any $x \in \widehat{G}_{adm,\lambda}$, let V_x be the corresponding Harish-Chandra module and let $I_x := Ann(V_x)$. The correspondence

$$\phi : \widehat{G}_{adm,\lambda} \to Prim(\mathfrak{g})_\lambda : x \longmapsto I_x$$

is often one-to-one, but generally speaking, several-to-one. We obtain by this correspondence a fairly fine partitioning of $\widehat{G}_{adm,\lambda}$. Towards the end of this talk, we shall show how the W-graph structure of a block allows one to determine exactly when two representations living in the same cell share a common primitive ideal.

3.2. **Nilpotent Orbits.** $U(\mathfrak{g})$ is naturally filtered according to

$$U^n(\mathfrak{g}) = \{X \in U(\mathfrak{g}) \mid X = \text{product of} \leq n \text{ elements of } \mathfrak{g}\}$$

and the corresponding graded algebra

$$gr(U(\mathfrak{g})) = \bigoplus_{n=0}^{\infty} U^n\left(\mathfrak{g}\right)/U^{n-1}(\mathfrak{g})$$

is well defined, and, in fact

$$\begin{aligned} gr\left(U\left(\mathfrak{g}\right)\right) &\approx S\left(\mathfrak{g}\right) := \text{the symmetric algebra of } \mathfrak{g} \\ &\approx \mathbb{C}[\mathfrak{g}^*] := \text{the ring of polynomials on } \mathfrak{g}^* \end{aligned}$$

Definition 3.2. *Let J be a primitive ideal and set*

$$\mathcal{V}(J) = \{\lambda \in \mathfrak{g}^* \mid \phi(\lambda) = 0 \quad \forall \phi \in gr(J)\}$$

The affine variety $\mathcal{V}(J)$ is called the associated variety *of J.*

Theorem 3.3 (Borho, Brylinski, Joseph)**.** *$\mathcal{V}(J)$ is a closed, G-invariant subset of $\mathfrak{g}^*$. In fact, $\mathcal{V}(J)$ is the Zariski closure of a single nilpotent orbit in $\mathfrak{g}^*$*

Definition 3.4. *Let $x \in HC_\lambda$ .The* ***nilpotent orbit attached to*** *x is the unique dense orbit $\mathcal{O}_x$ in $\mathcal{V}(Ann(x))$.*

Lemma 3.5. *If $x, y \in \widehat{G}_{adm,\lambda}$ belong to the same W-cell, then $\mathcal{O}_x = \mathcal{O}_y$.*

This is so because, roughly speaking, tensoring with $\mathfrak{g}$ only affects the contribution of lower order terms that are anyway annihilated by the gradation process. We remark though that different cells can share the same nilpotent orbit. We will describe below how one can utilize the W-graph structure of the cell to explicitly identify which nilpotent orbit is attached to the representations of cell.

4. Weyl group representations

One of the most remarkable developments in 1980's was the discovery of how the irreducible representations of Weyl group mediate the myriad of connections between representation theory, the classification theory of primitive ideals, nilpotent orbits. It should be no surprise then that the W-graph structure of a block should have something to do with Weyl group representations.

In fact, the Weyl group W, or rather the Iwasora-Hecke deformation of W, is the fundamental apparatus at play in the construction of the KLV-polynomials, which in turn allow one to explicity compute the W-graph structure of $\widehat{G}_{adm,\lambda}$. Indeed, it turns out that the Weyl group action on the free $\mathbb{Z}$-module $\mathbb{Z}\widehat{G}_{adm,\lambda}$ obtained by specializing the action of the Hecke algebra to $q = 1$ identifies with coherent continuation representation [20] of W on the Grothendieck group of $G_{adm,\lambda}$. This action induces a particular (in general reducible) W-representation on each cell which in turn has a unique *special* constituent σ_C occuring with multiplicity one. Moreover, it turns out that each of the following means of attaching a W-representation to a cell C leads to the same result:

(i) $C \longrightarrow$ cell representation $\xrightarrow{\textit{unique special constituent}} \sigma_C \in \widehat{W}$

(ii) $C \longrightarrow$ nilpotent orbit $\mathcal{O}_C \longrightarrow$ a nilpotent orbit $\mathcal{O}_C$ and a trivial local system on $\mathcal{O}_C$ $\xrightarrow{\textit{Springer correspondence}} \sigma_C \in \widehat{W}$

(iii) $C \longrightarrow \{\text{primitive ideal } I_x \mid x \in C\} \rightarrow W \cdot span_{\mathbb{C}}\,\{\text{Goldie rank polynomial } p_{I_x} \mid x \in C\} \longrightarrow \ \sigma_C \in \widehat{W}$

So it would seem that the ultimate arbitrator in all that follows is again the Weyl group. What might be a bit surprising is how easy it is to obtain the representation of W carried by a cell from its induced W-graph. Let $\mathcal{C}$ be the W-graph of a cell (obtained by restricting the W-graph of $\widehat{G}_{adm,\lambda}$ to one of its strongly connected components). Regarding the vertices of $\mathcal{C}$ as a basis for the free $\mathbb{Z}$-module $\mathbb{Z}\mathcal{C}$ we define an action of the reflection corresponding to a simple root α on $\mathbb{Z}\mathcal{C}$ by

$$T_\alpha x = \begin{cases} -x & \alpha \in \tau(x) \\ x + \sum_{y:\alpha\in\tau(y)} m_{y\to x} y & \alpha \notin \tau(x) \end{cases}$$

Here $m_{y\to x}y$ is the multiplicity of the edge $y \to x$ and $\tau(x)$ is the descent set of x. It turns out that these operators satisfy $T_\alpha \circ T_{alpha} = 1$ as well as the braid relations for the simple generators of W and so yield a representation of the Weyl group on $\mathbb{Z}\mathcal{C}$. As remarked above this representation coincides with the W-representation on the cell induced by the coherent continuation representation on the Grothendieck group of $G_{adm,\lambda}$. It is possible, using the above formula, to explicitly compute the trace of the action of a representative of each conjugacy class in W, thereby determine the character of the cell representation, and then to identify explicitly the representation of W carried by the cell by expressing this character as a sum of irreducibles. In fact, this simple method of computation is tractable even for the large cells of E_8 (which contain roughly 50,000 vertices). In an appendix to this lecture we provide a set of tables showing the cell representations so obtained for all the exceptional groups.

5. Attaching Nilpotent Orbits to Cells

Computing the cell representation, identifying the special constituent σ_C of the cell representation and then applying the inverse Springer correspondence is one way of attaching a nilpotent orbit to a cell. In this section we provide an alternative method that utilizes only the properties of special nilpotent orbits and the combinatorial properties of the W-graph of a cell.

We first note that having restricted attention to representations with regular integral infinitesimal character, only a special class of nilpotent orbits will arise as the associated varieties of annihilators of these representations. This class of nilpotent orbits (which are in fact called *special* nilpotent orbits can be characterized in several different ways (see Definition/Theorem ?? below).

Let $\mathcal{N}$ be the nilpotent cone of nilpotent elements in $\mathfrak{g}$ and let G_{ad} denote the (complex) adjoint group of $\mathfrak{g}$. We shall denote by $G_{ad}/\mathcal{N}$ the set of G_{ad}-orbits in $\mathcal{N}$ and we identify this set with the set of nilpotent orbits in $\mathfrak{g}^*$ via some fixed invariant non-degenerate bilinear on $\mathfrak{g}$). We note that the set $G_{ad}/\mathcal{N}$ carries a natural partial ordering

$$\mathcal{O}' \leq \mathcal{O} \quad \iff \quad \mathcal{O}' \subset \overline{\mathcal{O}} := \text{the Zariski closure of } \mathcal{O}$$

In [17], and a bit more generally, in [6] a *duality map* $d : G_{ad}/\mathcal{N} \to G_{ad}/\mathcal{N}$ is defined. This map has the following properties:

- d is order-reversing: $\mathcal{O}' \leq \mathcal{O} \quad \iff \quad d(\mathcal{O}) \leq (\mathcal{O}')$.
- d restricted to its image is an involution: $d \circ d \circ d = d$.

We note that for classical $\mathfrak{g}$, where the nilpotent orbits are parameterized by certain partitions of $n =$ dimension of the standard representation of $\mathfrak{g}$, the duality map restricted to its image corresponds to the operation that sends an orbit corresponding to partition $\mathbf{p}$ to the orbit corresponding to the transpose partition $\mathbf{p}^t$.

Definition/Theorem 5.1. *A nilpotent orbit $\mathcal{O}$ is* ***special*** *if any of the following (equivalent) properties holds:*

- *$\mathcal{O}$ is the dense orbit in the associated variety of a primitive ideal with regular integral infinitesimal character.*

- *$\mathcal{O}$ lies in the image of the Spaltenstein-Barbasch-Vogan duality map d.*
- *The irreducible representation of W corresponding to $\mathcal{O}$ and the trivial local system on $\mathcal{O}$ via the Springer correspondence is a special representation of the Weyl group in the sense of [Lustig].*

As noted above, because we have restricted attention to representations with regular integral infinitesimal character, the nilpotent orbit $\mathcal{O}_C$ attached to a cell C will be always be special nilpotent orbit. In order to explicitly identify the special nilpotent orbit attached to a particular cell we shall need to make use of the partial ordering of the special orbits, the duality map, and a particular subset of the special orbits; the Richardson orbits.

5.1. **Standard Levis and Richardson orbits.** Let Γ be a subset of the simple roots Π of $\mathfrak{d}(\mathfrak{g};\mathfrak{h})$. The corresponding *standard Levi subalgebra* $\mathfrak{l}_\Gamma$ is the Lie subalgebra of $\mathfrak{g}$ generated by the root subspaces $\mathfrak{g}_\alpha$, $\alpha \in \Gamma$, and the Cartan subalgebra $\mathfrak{h}$. Equivalently,

$$\mathfrak{l}_\Gamma = \mathfrak{h} + \sum_{\alpha \in \mathbb{Z}\Gamma} \mathfrak{g}_\alpha$$

Definition/Theorem 5.2. *Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ be a parabolic subalgebra of $\mathfrak{g}$ with Levi subalgebra $\mathfrak{l}$ and nilradical $\mathfrak{n}$, and let $\mathcal{O}_\mathfrak{l}$ be a nilpotent orbit in $\mathfrak{l}$. The nilpotent orbit **induced from** $\mathcal{O}_\mathfrak{l}$ is*

$$ind_\mathfrak{l}^\mathfrak{g}(\mathcal{O}_\mathfrak{l}) := \textit{ unique dense orbit in } G_{ad} \cdot (\mathcal{O}_\mathfrak{l} + \mathfrak{n})$$

*(The theorem part of this statement is that the orbit $ind_\mathfrak{l}^\mathfrak{g}(\mathcal{O}_\mathfrak{l})$ specified as above is well-defined.) When $\mathcal{O}_\mathfrak{l} = \mathbf{0}_\mathfrak{l}$ (the trivial orbit in $\mathfrak{l}$) then such an induced orbit is called a **Richardson orbit**. When $\mathfrak{l} = \mathfrak{l}_\Gamma$ is a standard Levi subalgebra and $\mathcal{O}_\mathfrak{l} = \mathbf{0}_{\mathfrak{l}_\Gamma}$, then $ind_{\mathfrak{l}_\Gamma}^\mathfrak{g}(\mathbf{0}_{\mathfrak{l}_\Gamma})$ will be called the **Richardson orbit corresponding to** Γ and denoted by $\mathcal{R}_\Gamma$.*

The trivial orbit is always special (in fact, the trivial orbit is always the dual of the principal nilpotent orbit and vice-versa). Moreover, by a theorem of Lusztig, induction preserves the property of "special-ness". And so Richardson orbits are always special nilpotent orbits. However, not all special orbits are Richardson. (We note, nevertheless, that for $\mathfrak{sl}(n,\mathbb{C})$ every nilpotent orbit is both special and Richardson, and that in general the existence of non-Richardson special orbits is related to the lack of injectivity of the duality map $d: G_{ad}\backslash\mathcal{N} \to G_{ad}\backslash\mathcal{N}$).

5.2. **The Spaltenstein-Vogan criterion.** We next note that from the explicit formula for the cell representation (cf. §5) that for any $x \in C$ the cell representation will always contain the sign representation of $W_{\tau(x)}$. Using this fact and the following result of Spaltenstein

Proposition 5.3 (Corollary 1 of [16])**.** *A special orbit $\mathcal{O}$ is contained in the closure of a Richardson orbit $\mathcal{R}_\Gamma$ if and only if the (special) W-representation attached to $(\mathcal{O},\mathbf{1})$ contains the sign representation of W_Γ. Here W_Γ is the reflection subgroup of W generated by the simple reflections s_α, $\alpha \in \Gamma$.*

David Vogan has proved the following criterion

Theorem 5.4. *Suppose C is a cell of Harish-Chandra modules and let $\mathcal{O}_C$ be the associated nilpotent orbit (cf. §4.2). Then*

$$\mathcal{O}_C \subset \overline{\mathcal{R}_-} \quad \textbf{iff} \quad \exists\, x \in C \ s.t.\ \Gamma \subset \tau(x)$$

Thus, the descent sets that occur amongst the representations in a cell C constain which Richardson orbits can contain $\mathcal{O}_C$ in their closures and vice-versa. However, because not every special orbit is Richardson, knowing which Richardson orbits contain a given cell orbit $\mathcal{O}$ in their closures is not sufficient to determine $\mathcal{O}$. However, the following is true:

Observation 5.5 (Vogan). Every special nilpotent orbit $\mathcal{O}$ is determined by the Richardson orbits $\mathcal{R}_\Gamma$ that contain $\mathcal{O}$ in their closures together with the Richardson orbits that contain $d(\mathcal{O})$ in their closures.

5.3. **Tau signatures.** I will now describe how the Spaltenstein-Vogan criterion and Vogan's observation can be put to work to explicity attach nilpotent orbits to W-cells.

First of all, the correspondence $\Pi \rightarrow G_{ad} \backslash \mathcal{N} : \Gamma \mapsto \mathcal{R}_\Gamma$ is hardly one-to-one. But by a classical result of Dynkin, conjugacy classes of Levi subalgebras in a semisimple Lie algebra $\mathfrak{g}$ (and hence Richardson orbits in $\mathfrak{g}$) are in a one-to-one correspondence with W-conjugacy classes in 2^Π, the set of subsets of Π. This motivates the following definition:

Definition 5.6. *Fix an ordering of the simple roots* Π *and let* $\Psi \subset 2^\Pi$ *be the collection*

$$\Psi = \{\Gamma \in \Pi \mid \Gamma \textit{ is the first in the lexicographical ordering of its } W\textit{-conjugacy class}\}$$

We call such Ψ *a set of* ***standard*** Γ*'s.*

By Dynkin's result any such Ψ is in a one-to-one correspondence with the set of Richardson orbits, and this in turn allows us to put the following partial order on Ψ:

$$\Gamma \leq \Gamma' \quad \Longleftrightarrow \quad \mathcal{R}_\Gamma \subseteq \overline{\mathcal{R}_{\Gamma'}}$$

Definition 5.7. *Let* $\mathcal{O}$ *be a special nilpotent orbit. The* ***tau signature*** *of* $\mathcal{O}$ *is the pair of subsets of* Ψ *defined by*

$$\tau_{sig}(\mathcal{O}) = \left(\min\left\{\Gamma \in \Psi \mid \mathcal{O} \subset \overline{\mathcal{R}_\Gamma}\right\} \ , \ \min\left\{\Gamma \in \Psi \mid d(\mathcal{O}) \subset \overline{\mathcal{R}_\Gamma}\right\}\right)$$

It follows from Vogan's observation that a special orbit is completely determined by its tau signature.

Next,

Definition 5.8. *Let* C *be an* W*-cell and set*

$$\tau(C) = \{\tau(x) \mid x \in C\}$$
$$\tau^\vee(C) = \{\Pi - \tau(x) \mid x \in C\}$$

The ***tau signature*** *of* C *is the pair of subsets of* Ψ *defined by*

$$\tau_{sig}(C) = \{\min(\tau(C) \cap \Psi) \ , \ \min(\tau^\vee(C) \cap \Psi)\}$$

In terms of these tau signatures, we now have the following criterion:

Criterion 5.9. *Let* C *be an* W*-cell and let* $\mathcal{O}$ *be a special nilpotent orbit. Then*

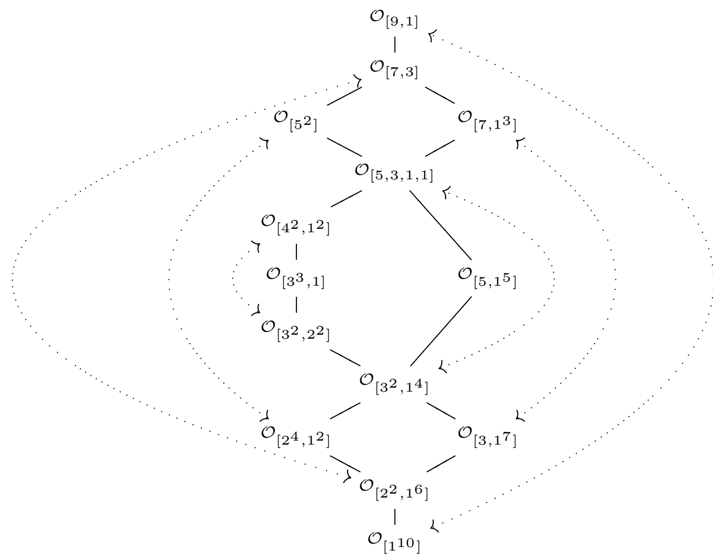
$$\mathcal{O}_C = \mathcal{O} \quad \Longleftrightarrow \quad \tau_{sig}(C) = \tau(\mathcal{O}) \ .$$

(That $\tau^\vee(C)$ is the correct counterpart to $\left\{\Gamma \in \Psi \mid d(\mathcal{O}) \subset \overline{\mathcal{R}_\Gamma}\right\}$ comes from the fact that the $\tau(C)$ is a collection of descent sets for an W-cell for a block of $\mathbb{G}$ then $\tau^\vee(C)$ will be a collection of descent sets for a cell in a block of $\mathbb{G}^\vee$ and the fact that this correspondence is consistent with the Barbasch-Vogan duality map $d : G_{ad} \backslash \mathcal{N}_\mathfrak{g} \rightarrow G_{ad}^\vee \backslash \mathcal{N}_{\mathfrak{g}^\vee}$).

Let me now give an example showing how easy it is to apply this criterion.

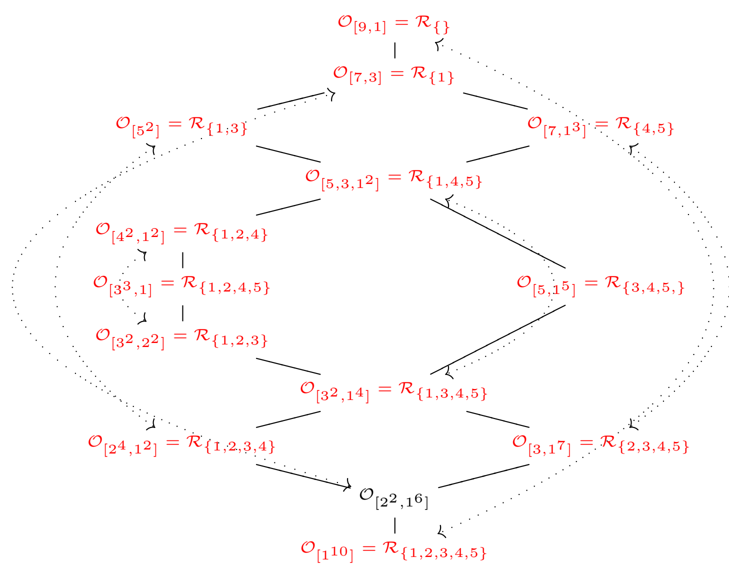
Below is the Hasse diagram for the special orbits of $D_5 \approx \mathfrak{so}(10, \mathbb{C})$, with the orbits labeled by the corresponding partitions of 10. The dotted lines indicate the action of the duality map d:

The Special Orbits of D_5

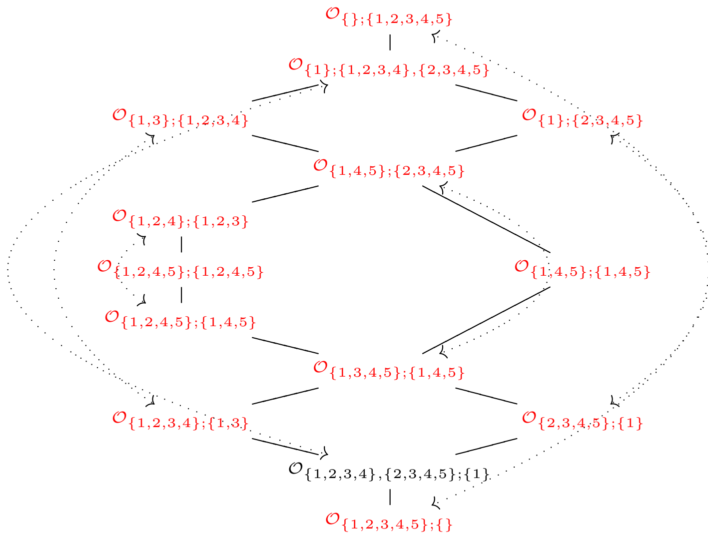


Below is the same diagram, except that we explicitly identify which of these special orbits is Richardson.

Richardson Orbits of D_5



From the last diagram, we can easily read off the minimal Richardson orbits that contain a given special orbit, which Richardsonson orbits contain the dual of that orbit, and then use that information to ascribe tau signatures to the special orbits. This yields



Next, we use the Atlas program to compute the W-graph of the big $(SO(5,5)\times SO(5,5)\ )$ block of D_5. This yields a table like

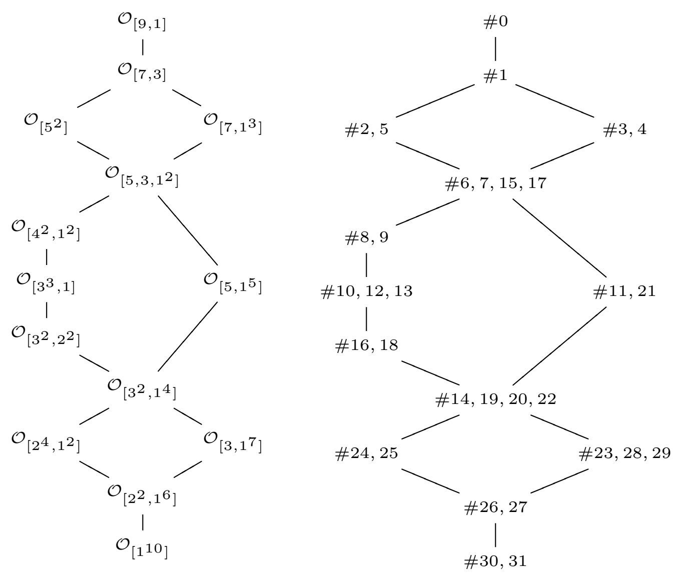
```
// Individual cells.
// cell #0:
0[0]: {}

// cell #1:
0[1]: {2} --> 1,2
1[3]: {1} --> 0
2[5]: {3} --> 0,3,4
3[13]: {5} --> 2
4[14]: {4} --> 2
  *
  *
  *
// cell #29:
0[328]: {1,2,4,5} --> 2,3
1[340]: {2,3,4,5} --> 2
2[358]: {1,3,4,5} --> 0,1
3[364]: {1,2,3} --> 0

// cell #30:
0[353]: {1,2,3,4,5}

// cell #31:
0[357]: {1,2,3,4,5}
```

From this table, and the partial ordering of the standard Γ's (from Figure) we can readily identify the tau signature of each cell of the block.

```
cell #     tau signature
    0      {} , {1,2,3,4,5}
    1      {1} , {1,2,3,4}
    2      {1} , {2,3,4,5}
    3      {1,3} , {1,3,4,5}
    *      *
    *      *
    *      *
   28      {2,3,4,5} , {1}
   29      {2,3,4,5} , {1}
   30      {1,2,3,4,5} , {}
   31      {1,2,3,4,5} , {}
```

Comparing the tau signatures of the special orbits with those of the cells we obtain the following cell-orbit correspondences

Cell-Orbit Correspondences for $SO(5,5)$



Note that several W-cells may correspond to the same nilpotent orbit. This means, in particular, that annihilators of each of the representations in cells share the same associated variety. In this fashion, Atlas provides not only a means for enumerating the representations in $\widehat{G}_{adm,\lambda}$, but also means to explicitly attach nilpotent orbits to these representations.

6. Partitioning $\widehat{G}_{adm,\lambda}$ into subsets sharing the same primitive ideal

In the preceding section, we showed how the W-graph of a cell, in fact, just the set of vertex weights, allowed us to figure out which nilpotent orbit should be attached to the representations in the cell. In this section I shall so how the W-graph structure allows us to determine when two representations share the same primitive ideal.

First, however, I will quickly review some apparatus from of the theory of primitive ideals.

6.1. **Organization of** $Prim(\mathfrak{g})_\lambda$**.** Irreducible Harish-Chandra modules are the $U(\mathfrak{g}$-modules that most naturally in the study of admissible representations of a real reductive group. However, the most convenient setting for discussing primitive ideals is in the category of highest weight modules. We begin by recalling a basic construction of the simple highest weight modules.

Definition 6.1. *Let* $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ *be a Borel subalgebra of* $\mathfrak{g}$ *and let* $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{h})}\alpha$*. For any* $\lambda \in \mathfrak{h}^*$ *let* $M(\lambda)$ *denote the Verma module of highest weight* $\lambda - \rho$*; that is to say,* $M(\lambda)$ *is the left* $U(\mathfrak{g})$*-module*

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$$

Theorem 6.2. *Let* $\lambda \in \mathfrak{h}^*$*. Then*

(i) The Verma module $M(\lambda)$ *has a unique irreducible quotient module* $L(\lambda)$ *which is of highest weight* $\lambda-\rho$*.*

(ii) Every irreducible highest weight module is isomorphic to some $L(\lambda)$*.*

Theorem 6.3 (Duflo)**.** *For* $w \in W(\mathfrak{g},\mathfrak{h})$ *let*

$$L_w = \textit{unique irreducible quotient of } M(-w\rho)$$

Then

$$\varphi : W \to Prim\,(\mathfrak{g})_\rho : w \to Ann\,(L_w)$$

is a surjection.

(Recall that $Prim\,(\mathfrak{g})_\rho$ is the set of primitive ideals in $U(\mathfrak{g})$ with infinitesimal character ρ.)

In view of Duflo's theorem, parameterizing $Prim\,(\mathfrak{g})_\rho$ is tantamount to understanding the fiber of the map $\varphi : W \to Prim\,(\mathfrak{g})_\rho$. We have a special name for these fibers:

Definition 6.4. *Let $\sim$ be the equivalence relation on W defined by*

$$w \sim w' \quad \Longleftrightarrow \quad Ann(L_w) = Ann(L_{w'})$$

The corresponding equivalence classes of elements of W are called ***left cells*** *in W.*

We can also collect together the elements of W which lead to the same nilpotent orbit. Before doing this formally, we note that because the simple highest weight modules L_w all have infinitesimal character ρ (actually $W \cdot \rho$) which is always regular and integral, the corresponding nilpotent orbits will always be special nilpotent orbits, via a theorem of Barbasch and Vogan ([6]). Let $\mathcal{S}$ denote the set of special nilpotent orbits. It turns out that the map $W \to \mathcal{S}$ given by

$$w \mapsto \mathcal{O}_w \equiv \text{the unique dense orbit in the associated variety of } Ann\,(L_w)$$

is also onto.

Definition 6.5. *Let $\approx$ be the equivalence relation on W defined by*

$$w \approx w' \quad \Longleftrightarrow \quad \mathcal{O}_{Ann(L_w)} = \mathcal{O}_{Ann(L_{w'})}$$

The corresponding equivalence classes of elements of W are ***double cells*** *in W. For any special orbit $\mathcal{O} \in \mathcal{S}$ we denote by $\mathcal{C}_\mathcal{O}$ the corresponding double cell in W.*

Theorem 6.6 (Barbasch-Vogan, Joseph)**.**

(i) *The decomposition of W into left cells is refinement of its partitioning into double cells so that*

$$W = \coprod_{\mathcal{O} \in \mathcal{S}} \left(\coprod_{left\ cells\ \ell \subset \mathcal{C}_\mathcal{O}} \ell \right)$$

(ii) *Let $\mathcal{O}$ be a special orbit and let $\sigma_\mathcal{O}$ be the corresponding special representation of the Weyl group provided by the Springer correspondence. Then the number of left cells in $\mathcal{C}_\mathcal{O}$ is exactly the same as the dimension of $\sigma_\mathcal{O}$.*

In particular,

$$\left| Prim\,(\mathfrak{g})_\rho \right| = \sum_{\mathcal{O} \in \mathcal{S}} \dim \sigma_\mathcal{O}$$

We remark that via the Borho-Janzten translation principle this description of $Prim\,(\mathfrak{g})_\rho$ in terms of special nilpotent orbits and the corresponding special representations of W carries over, in exactly the same way, for $Prim\,(\mathfrak{g})_\lambda$ whenever λ is regular and integral.

6.2. **Two pictures at inf char ρ.** The above discussion can be summarized by following diagram HW-modules

W	$\{L_w \mid w \in W\}$	same inf char
$\cup$	$\cup$	
$\mathcal{C} : dbl\ cell$	$\{L_w \mid w \in \mathcal{C}\}$	same nilpotent orbit
$\cup$	$\cup$	
$\ell : left\ cell$	$\{L_w \mid w \in \ell\}$	same primitive ideal

Regarding W simply as a parameter set for the set of simple highest weight modules of infinitesimal character ρ, what we seek to develop next is an analogous picture for the set of irreducible admissible representations in a block of $\widehat{G}_{adm,\rho}$ as indexed by Atlas's block command.

B : block of HC-modules	$\{\pi_x \mid x \in B\}$	same inf char
$\cup$	$\cup$	
C : cell of HC-modules	$\{\pi_x \mid x \in C\}$	same nilpotent orbit
$\cup$	$\cup$	
?	$\{\pi_x \mid x \in ?\}$	same primitive ideal

We shall show next how the W-graph structure of a cell enables us to figure out the analog of a left cell sitting inside an W-cell. To do that we need to explain first a particular invariant of primitive ideals and its relation to the descent sets of the representations in a cell.

6.3. **Tau invariants.** Let L_w again denote the simple highest weight module of highest weight $-w\rho - \rho$ and let $I_w := Ann\,(L_w)$. We have

- I_{w_o} : unique max ideal (augmentation ideal, annihilator of triv rep)
- I_e = unique min PI at inf char ρ ($\leq$ by inclusion)
- I_{s_α}, $\alpha \in \Pi$: "pen-minimal" ideals

A little more precisely, by this last fact we mean,

Theorem 6.7. *The primitive ideals I_{s_α}, $\alpha \in \Pi$, are all distinct from each other and I_e. Any primitive ideal strictly containing I_e contains at least one of the I_{s_α}.*

which in turn leads us to the following definition:

Definition 6.8. *The **tau invariant** of a primitive ideal I containing I_e is*

$$\tau(I) = \{\alpha \in \Pi \mid I_{s_\alpha} \subset I\}$$

The following theorem is the key to what follows. (It also justifies our notation for descent sets).

Theorem 6.9 (Vogan)**.** *Let x be an element of a cell C of HC modules and let $\tau(x)$ be its descent set (as, for example, obtained from the W-graph of C). Then*

$$\tau(x) = \textit{tau-invariant of } Ann(x)$$

6.4. **A partitioning of W-cells.** Recall the the W-graph of cell C attaches to each for each irreducible admissible representation $x \in C$

- a vertex index i
- a tau invariant $\tau\,[i]$ = tau invariant of $Ann\,(x)$
- a list of edges with multiplicities $e\,[i] = [(j_1, m_1)\,, (j_2, m_d)\,, \ldots, (j_k, m_k)]$

With the goal of figuring out which irreducible admissible representations share the same primitive ideal, the first thing one might do is collect together the cell elements which share the same tau invariant (descent set). We shall call such collections τ_0**-subcells** and write

$$x \sim_{\tau_0} y \quad \Longleftrightarrow \quad \tau(x) = \tau(y)$$
$$C = \coprod_{[x]_0 \in C/\sim_{\tau_0}} [x]_0$$

We note that a τ_0-subcell corresponds a collection of admissible representations that share the same nilpotent orbit $\mathcal{O}_C$ and a common tau invariant.

Next we define τ_1**-subcells** by setting $\tau_1(x) = \{\tau(y) \mid x \to y \text{ is an edge}\}$ and writing

$$x \sim_{\tau_1} y \quad \Longleftrightarrow \quad \tau(x) = \tau(y) \text{ and } \tau_1(x) = \tau_1(y)$$
$$C = \coprod_{[x]_1 \in C/\sim_{\tau_1}} [x]_1$$

Similarly, $\boldsymbol{\tau_2}$ **subcells** are defined by setting $\tau_2(x) = \{\tau_1(y) \mid x \to y \text{ is an edge}\}$

$$x \sim_{\tau_2} y \quad \Longleftrightarrow \quad \tau_0(x) = \tau_0(y)\ ,\ \tau_1(x) = \tau_1(y)\ ,\ \text{and } \tau_2(x) = \tau_2(y)$$
$$C = \coprod_{[x]_2 \in C/\sim_{\tau_2}} [x]_2$$

We can clearly continue in this fashion, obtaining at each new iteration a refinement of the previous partitioning of the cell. Since the cells are finite sets, however, eventually this iterative partitioning scheme must stabilize with τ_j-subcells coinciding with τ_{j+1}-subcells for all sufficiently large j. We shall refer to the ultimate stable partition of the cell as its τ_∞-partitioning and write

$$C = \coprod_{[x]_\infty \in C/\sim_{\tau_\infty}} [x]_{\tau_\infty}$$

Lemma 6.10. *The τ_∞ partitioning of a cell of HC-modules is compatible with the partitioning of the cell into subcells consisting of representations with the same primitive ideal:*

$$Ann(x) = Ann(y) \quad \Longrightarrow \quad x \text{ and } y \text{ live in same } \tau_\infty\text{-subcell.}$$

This follows from well-definedness of Translation Functor for primitive ideals.

These τ_∞ invariants are essentially the same as the *generalized τ-invariants* introduced by Vogan in [V1]. Also in that paper is the following, by now long-standing, conjecture:

Conjecture 6.11 (Vogan, 1979)**.** *The generalized τ-invariants completely split the set $Prim(\mathfrak{g})_\lambda$; that is to say, if the generalized τ-invariants of two primitive ideals $J, J' \in Prim(\mathfrak{g})_\lambda$ coincide then $J = J'$.*

Vogan proved this for type A_n subalgebras in [V1]. Later, in the 1990's Devra Garfinkle confirmed Vogan's conjecture for type B_n and C_n ([D1], [D2], [D3]).

Theorem 6.12. *Let C be any cell in any real form of any exceptional group G. Then the τ_∞ partitioning of C coincides precisely with the partitioning of the cell into sets of irreducible admissible representations sharing the same primitive ideal:*

$$x \sim_\infty y \quad \Longleftrightarrow \quad Ann(x) = Ann(y)$$

The proof of this theorem is by direct computation. Using the Atlas software we have explicitly computed the W-graph every such cell. It is then a simple matter to write a program that uses the W-graph data of a cell to partition it into a collection τ_∞-subcells. What one finds is that for each exceptional Lie group $G_\mathbb{R}$ and each cell of representations of a $G_\mathbb{R}$ one has

$$\#P_\infty\text{-subcells} = \dim \text{ special } W\text{-rep attached to cell}$$

But the dimension of the special representation attached to a cell is also the number of distinct primitive ideals with associated nilpotent orbit $\mathcal{O}_C$. Since the τ_∞-partitioning scheme is compatible with the partitioning by common primitive ideals, and so is at worse a coarsening of the partitioning by primitive ideals, we must conclude the τ_∞-partitioning scheme coincides with the partitioning by primitive ideals; as

$$\#P_\infty\text{-subcells} = max\#\text{primitive ideals in cell}$$

7. Algorithms and Results

For more details on the algorithms used in the W-graph computations described in this lecture, as well as tables of the results for the exceptional groups, we refer the interested reader to [B].

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9. Sources

Information about the Atlas project, expository notes, and the Atlas software can be found on the Atlas for Lie Groups web site: http:\\atlas.math.umd.edu

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