W-Graphs, Nilpotent Orbits and Primitive Ideals

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General Setting

G : a real reductive Lie group arising as the set of real points of a connected, linear, complex reductive group $G_{\mathbb{C}}$ defined over $\mathbb{R}$

$\widehat{G}_{adm} = \{\text{irr adm reps of } G\}$

Objective: an explicit partitioning of $\widehat{G}_{adm}$ via algebraic invariants.

- $\widehat{G}_{adm} \longrightarrow \widehat{G}_{adm,\lambda}$, reps with fixed inf char λ
- $\widehat{G}_{adm,\lambda} \longrightarrow$ Associated Variety / Nilpotent Orbits
- $\widehat{G}_{adm,\lambda,\mathcal{O}} \longrightarrow$ reps sharing same primitive ideal

$\Longrightarrow$ a complete "phenomenology" of $\widehat{G}_{adm}$.

Bullet Point: Atlas software actually makes this program not only tractable but completely explicit.

Reduction to Inf Char ρ

Schur's lemma $\rightarrow$ the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts by scalars on $x \in \widehat{G}_{adm}$

Harish-Chandra homomorphism $\phi_{HC} : Z(\mathfrak{g}) \longleftrightarrow S(\mathfrak{h})^W$

$\rightarrow$ action of $Z(\mathfrak{g})$ on x can be characterized by an *infinitesimal character* $\lambda_x \in \mathfrak{h}^* \backslash W$

Let $\widehat{G}_{adm,\lambda} := \{\text{irr. adm. reps with inf char } \lambda\}$

Then (Thm. Harish-Chandra)

$$\left| \widehat{G}_{adm,\lambda} \right| < \infty$$

and

$$\widehat{G}_{adm} \quad = \quad \coprod_{\lambda \in \mathfrak{h}^*/W} \widehat{G}_{adm,\lambda}$$

Borho-Jantzen-Zuckerman translation principle : structure of $\widehat{G}_{adm,\lambda}$ is consistent on entire translation family of $\widehat{G}_{adm,\lambda}$

Largest, most comprehensive family: translation family of trivial rep

Assumption: λ is assumed to be regular and integral $\Longrightarrow \widehat{G}_{adm,\lambda} \sim \widehat{G}_{adm,\rho}$

Let

$$HC_\lambda := \left\{ V_{K\text{-finite}} \mid V \in \widehat{G}_{adm,\lambda} \right\}$$

Definition

Given two reps x, y in HC_λ, we say

$$x \rightsquigarrow y \quad \Longleftrightarrow \quad \exists \text{ f.d. rep } F \subset \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes^n} \quad \text{s.t.} \quad x \text{ occurs as subquotient of } y \otimes F$$

and

$$x \sim y \;\text{ if } x \rightsquigarrow y \text{ and } y \rightsquigarrow x$$

The equivalence classes for the relation $\sim$ are called **cells** (of HC-modules).

Blocks and Cells of HC modules: W-graph formulation

Definition

Given $x, y \in HC_\lambda$, we write

$$x \xrightarrow{\ m\ } y \quad \Longrightarrow \quad x \text{ occurs with multiplicity } m \text{ in } y \otimes \mathfrak{g}$$

The relation "$\rightarrow$" gives HC_λ the structure of a directed graph.

$$\begin{array}{rcl} \text{“} \rightsquigarrow \text{”} & \longleftrightarrow & \text{transitive closure of “} \rightarrow \text{”} \\ \text{cells of reps} & \longleftrightarrow & \text{strongly connected components of graph} \\ \text{blocks of reps} & \longleftrightarrow & \text{connected components of graph} \end{array}$$

The `atlas` software explicitly computes this digraph structure as a *by-product* of its computation of the *KLV*-polynomials.

W-graphs

For each $x, y \in HC_\lambda$ Atlas computes a KLV polynomial $P_{x,y}(q)$.

Definition

The ***W*-graph** of HC_λ is the weighted digraph where:

- the **vertices** are the elements $x \in HC_\lambda$
- there is an **edge** $x \longrightarrow y$ of **multiplicity** m between two vertices if

$$\text{coefficient of } q^{(|x|-|y|-1)/2} \text{ in } P_{x,y}(q) = m \neq 0$$

- there is assigned to each vertex x a subset $\tau(x)$ of the set of simple roots of $\mathfrak{g}$, the **descent set** of x.

Digression: Blocks of irreducible Harish-Chandra modules in Atlas

"Under the hood" of the `atlas` software is a parameterization of $\widehat{G}_{adm,\rho}$ in terms of pairs

$$(x, y) \in K\backslash G/B \times K^\vee\backslash G^\vee/B^\vee$$

Definition

A **block** of representations is set of representations for which the pairs (x, y) range over $K\backslash G/B \times K^\vee\backslash G^\vee/B^\vee$ corresponding to fixed real forms of G and $G^\vee$.

Atlas's computations take place block by block.

(Atlas blocks are still the connected components of the W-graph of HC_ρ)

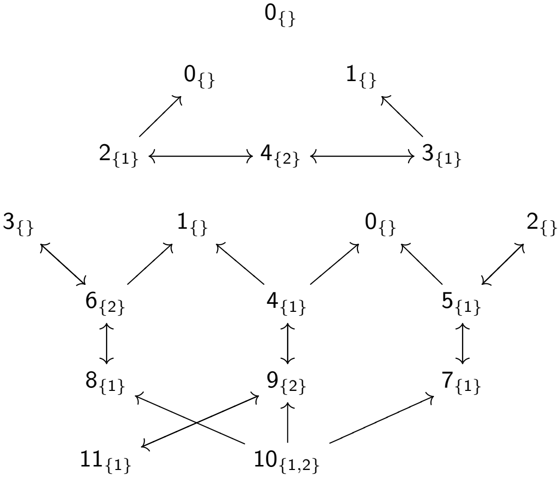
Example: $Sp(4, R)$

$\widehat{G}_{adm,\rho}$ has 18 distinct irreducibles

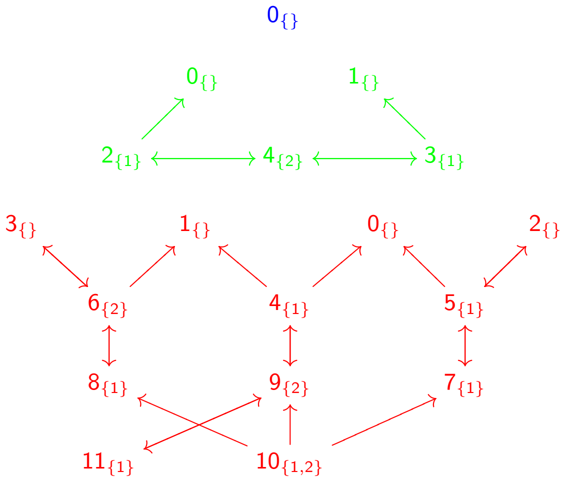
$Sp(4, \mathbb{C})^{\vee} \approx PSO(5, \mathbb{C})$ has three real forms: $PSO\,(5)$, $PSO\,(4,1)$, $PSO\,(3,2)$
$Sp(4, \mathbb{R}$ has three blocks of representations.

	$PSO\,(5)$	$PSO\,(4,1)$	$PSO\,(3,2)$
$Sp\,(4,\mathbb{R})$	1	5	12

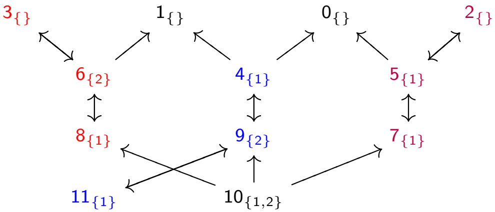
W-graph for $Sp(4, R)$



W-graph for $Sp(4, R)$



Cells in the $Sp(4,R) \times PSO(3,2)$ block of $Sp(4,C)$



#	(x,y)	*CSA*	Π	τ	*Properties*
0	$(0,6)$	$\mathbb{T}^2$	$[i1, i1]$	$\{\}$	non-hol DS
1	$(1,6)$	$\mathbb{T}^2$	$[i1, i1]$	$\{\}$	non-hol DS
2	$(2,6)$	$\mathbb{T}^2$	$[ic, i1]$	$\{1\}$	anti-hol DS
3	$(3,6)$	$\mathbb{T}^2$	$[ic, i1]$	$\{1\}$	hol DS
4	$(4,5)$	$\mathbb{C}^\times$	$[r1, C+]$	$\{1\}$	non-tempered, unitary, not HW
5	$(5,4)$	$\mathbb{T}\mathbb{R}^\times$	$[C+, r1]$	$\{2\}$	non-tempered, unitary, LW
6	$(6,4)$	$\mathbb{T}\mathbb{R}^\times$	$[C+, r1]$	$\{2\}$	non-tempered, unitary, HW
7	$(7,3)$	$\mathbb{T}\mathbb{R}^\times$	$[C-, i1]$	$\{1\}$	non-unitary, LW
8	$(8,3)$	$\mathbb{T}\mathbb{R}^\times$	$[C-, i1]$	$\{1\}$	non-unitary, HW
9	$(9,2)$	$\mathbb{C}^\times$	$[i2, C-]$	$\{2\}$	???
10	$(10,0)$	$(\mathbb{R}^\times)^2$	$[r2, r1]$	$\{1,2\}$	trivial rep
11	$(10,1)$	$(\mathbb{R}^\times)^2$	$[r2, rn]$	$\{1\}$	PS w/ LKTs [1,1],[-1,-1]

Blocks of $E_8 \sim E_8^\vee$

	e_8	$E_8(e_7, su(2))$	$E_8(\mathbb{R})$
e_8	0	0	1
$E_8(e_7, su(2))$	0	3150	73410
$E_8(\mathbb{R})$	1	73410	453060

$$1 + 73410 + 453060 = 526,471 \text{ admissible reps in } \widehat{G}_{adm,\rho} \text{ for } E_8$$

No corresponding Atlas $\# \leftrightarrow$ rep-type dictionary for E_8 (or general G for that matter)

Original Motivation: organize Atlas output in terms of humanly recognizable algebraic invariants

Definition

Let V be an irreducible $U(\mathfrak{g})$-module.

$$Ann(V) := \{X \in U(\mathfrak{g}) \mid Xv = 0 \quad , \quad \forall\, v \in V\}$$

is a two-sided ideal in $U(\mathfrak{g})$. It is called the **primitive ideal** in $U(\mathfrak{g})$ attached to V.

Fact: $Ann(V) = Ann(V') \qquad \Longrightarrow \qquad inf\ ch\ V = inf\ ch\ V'$

Def: $Prim\,(\mathfrak{g})_\lambda$:= set of primitive ideals in $U(\mathfrak{g})$ with inf char λ

The correspondence

$$HC_\lambda \to Prim\,(\mathfrak{g})_\lambda : x \longmapsto Ann(x)$$

is often one-to-one, but generally speaking, several-to-one.

$\Rightarrow$ a fairly fine grained-partitioning of HC_λ

Nilpotent Orbits

$U(\mathfrak{g})$ is naturally filtered according to

$$U^n(\mathfrak{g}) = \{X \in U(\mathfrak{g}) \mid X = \text{product of} \leq n \text{ elements of } \mathfrak{g}\}$$

The graded algebra

$$gr(U(\mathfrak{g})) = \bigoplus_{n=0}^{\infty} U^n(\mathfrak{g}) / U^{n-1}(\mathfrak{g})$$

is well defined, and, in fact

$$gr(U(\mathfrak{g})) \approx S(\mathfrak{g})$$

Definition

Let J be a primitive ideal and set

$$\mathcal{V}(J) = \{\lambda \in \mathfrak{g}^* \mid \phi(\lambda) = 0 \quad \forall \phi \in gr(J)\}$$

The affine variety $\mathcal{V}(J)$ is called the **associated variety** of J.

Theorem

$\mathcal{V}(J)$ is the Zariski closure of a single nilpotent orbit in $\mathfrak{g}^*$

Definition

Let $x \in HC_\lambda$. The **nilpotent orbit attached to** x is the unique dense orbit $\mathcal{O}_x$ in $\mathcal{V}(Ann(x))$.

Lemma

If x, y belong to the same cell of HC-modules then $\mathcal{O}_x = \mathcal{O}_y$.

(assoc variety doesn't change after tensoring with a finite-dim rep)

Different cells can share the same nilpotent orbit

$\rightsquigarrow$ rather coarse invariant of HC-modules

Cell Representations

W acts naturally on the Grothendieck group $\mathbb{Z}HC_\lambda$ via the "coherent continuation representation"

The W-representation carried by a cell is encoded in its W-graph.

The action of a simple reflection on cell rep corresponds to

$$T_i x = \begin{cases} -x & i \in \tau(x) \\ x + \sum_{y : i \in \tau(y)} m_{y \to x} y & i \notin \tau(x) \end{cases}$$

The W-rep carried by a cell can be computed by evaluating

$$\chi_C(s_i \cdots s_j) = trace\,(T_i \cdots T_j)$$

on a representative $s_i \cdots s_j$ of each conjugacy class and then decomposing this character into a sum of irreducible characters (i.e., brute force is feasible)

Or via branching rules (Jackson-Noel) (spotting occurence of sign reps of Levi subgroups)

Cells $\longrightarrow$ Nilpotent Orbits: Notation / Apparatus

$\mathfrak{g} = Lie\,(G_{\mathbb{R}})_{\mathbb{C}}$; $\quad \mathfrak{h}$, a CSA for $\mathfrak{g}$;
$\Delta = \Delta\,(\mathfrak{h}, \mathfrak{g})$, roots of $\mathfrak{h}$ in $\mathfrak{g}$;
$\Pi \subset \Delta$, choice of simple roots in Δ;

G : adjoint group of $\mathfrak{g}$

$\mathcal{N}_{\mathfrak{g}}$: nilpotent cone in $\mathfrak{g} \sim \mathfrak{g}^*$

$\mathcal{S} \equiv \{$special nilpotent orbits$\} \subset \mathcal{N}_{\mathfrak{g}}$

$\mathcal{S} \equiv \{$assoc varieties of prim ideals of reg int inf char$\}$

or

$\mathcal{S} \equiv \{$orbits that map to Lusztig's special W-reps via Springer correspondence$\}$

or

Let $d : G\backslash\mathcal{N}_{\mathfrak{g}} \rightarrow G\backslash\mathcal{N}_{\mathfrak{g}}$: the Spaltenstein-Barbasch-Vogan duality map

$\mathcal{S} \equiv image(d)$

d restricts to an order-reversing involution on $\mathcal{S}$

Digression: Standard Levis and Richardson orbits

Definition

Let Γ be a subset of the simple roots. The corresponding **standard Levi subalgebra** $\mathfrak{l}_\Gamma$ is the subalgebra

$$\mathfrak{l}_\Gamma = \mathfrak{h} + \sum_{\alpha \in \mathbb{Z}\Gamma} \mathfrak{g}_\alpha$$

Definition

Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ be the Levi decomposition of a parabolic subalgebra of $\mathfrak{g}$ and let $\mathcal{O}$ be nilpotent orbit in $\mathfrak{l}$.

$$ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}) := \text{unique dense orbit in } G \cdot (\mathcal{O} + \mathfrak{n})$$

When $\mathcal{O} = \mathbf{0}_{\mathfrak{l}}$, $ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O})$ is called the **Richardson orbit** corresponding to $\mathfrak{l}$.
When $\mathfrak{l} \sim_G \mathfrak{l}_\Gamma$, $\Gamma \subset \Pi$ we shall write shall write

$$\mathcal{R}_\Gamma = ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}(\mathbf{0}_{\mathfrak{l}_\Gamma})$$

for the corresponding Richardson orbit.

Good:

Richardson orbits are always special orbits.

Orbits attached to $x \in \widehat{G}_{adm,\rho}$ are always special orbits.

Richardson orbits are parameterized by subsets of simple roots (a possible point of contact with Atlas computations)

Unfortunate: special orbits are not always Richardson orbits

Theorem (Spaltenstein, Vogan)

Suppose C is a cell of H-C modules with associated special nilpotent orbit $\mathcal{O}_C$ and let $\Gamma \subseteq \Pi$. Then

$$\mathcal{O}_C \subset \overline{\mathcal{R}_\Gamma} = \overline{ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}\left(\mathbf{0}_{\mathfrak{l}_\Gamma}\right)} \quad \Longleftrightarrow \quad \exists\, x \in C \;\; s.t.\, \Gamma \subset \tau(x)$$

Upshot: τ-invariants of a cell C constrain which Richardson orbit closures can contain $\mathcal{O}_C$

Set

$$\tau(C) \equiv \{\tau(x) \mid x \in C\} = \{\tau\text{-invariants of reps in } C\}$$

Is it plausible that $\tau(C)$ identifies $\mathcal{O}_C$?

Observation: # distinct $\tau(C)$ = # special nilpotent orbits

Using Richardson orbits to "frame" special orbits

For classical $\mathfrak{g}$ every special orbit is either Richardson or the intersection of two Richardsons. Not so for exceptional.

Fact: For $\mathfrak{g}$ simple, every special orbit $\mathcal{O}$ **is** determined by

(i) the Richardson orbits that contain $\mathcal{O}$

(ii) the Richardon orbits that contain $d(\mathcal{O})$

David's Insight: The tau invariants of a cell should tell us which Richardson orbits contain $\mathcal{O}_C$ and which Richardson orbits contain $d\left(\mathcal{O}_C\right)$.

Special Apparatus: tau signatures

$\Psi = 2^{\Pi} :=$ { subsets of simple roots}

Partial Order Ψ as follows:

$$\Gamma \leq \Gamma' \Longleftrightarrow ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}(\mathbf{0}) \subseteq \overline{ind_{\mathfrak{l}_{\Gamma'}}^{\mathfrak{g}}(\mathbf{0})}$$

Remark: this ordering tends to reverse the natural ordering of Ψ by set inclusion.

$$\text{trivial orbit} = ind_{\mathfrak{g}}^{\mathfrak{g}}(\mathbf{0}) = \mathcal{R}_{\Pi} \subset \cdots\cdots \subset \mathcal{R}_{\{\}} = ind_{\mathfrak{h}}^{\mathfrak{g}}(\mathbf{0}_{\mathfrak{h}}) = \text{principal orbit}$$

Definition: The **tau signature** of an W-cell C is the pair

$$\tau_{sig}(C) \equiv \left(min\left(\tau(C) \cap \Psi\right) \ , \ min\left(\tau^{\vee}(C) \cap \Psi\right)\right)$$

Here $\tau^{\vee}(C)$ is the set of Π-complements of tau invariants in C:

$$\tau^{\vee}(C) = \{\Pi - \tau(x) \mid x \in C\}$$

Tau signatures for special orbits

Definition: Let $\mathcal{O}$ be a special orbit. The **tau signature** of $\mathcal{O}$ is the pair $(\tau(\mathcal{O}), \tau^\vee(\mathcal{O}))$ where

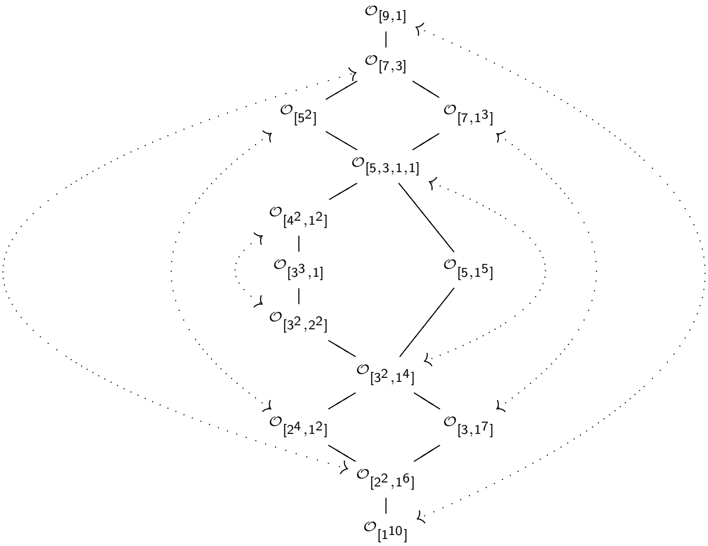
$$\tau(\mathcal{O}) = min\left\{\Gamma \in \Psi \mid \mathcal{O} \subset \overline{ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}(\mathbf{0}_{\mathfrak{l}_\Gamma})}\right\}$$
$$\tau^\vee(\mathcal{O}) = min\left\{\Gamma \in \Psi \mid d(\mathcal{O}) \subset \overline{ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}(\mathbf{0}_{\mathfrak{l}_\Gamma})}\right\}$$

The point: we are using pairs of subsets of simple roots to tell us when a Richardson orbit closure can contain a special orbit (or its dual).

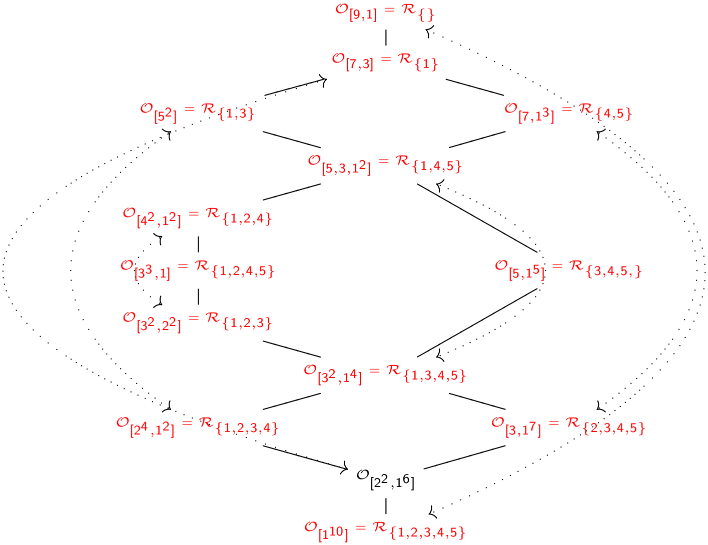
Corollary (to S-V criterion)

$$\mathcal{O}_C = \mathcal{O} \quad \Longleftrightarrow \quad \tau_{sig}(C) = \tau_{sig}(\mathcal{O})$$

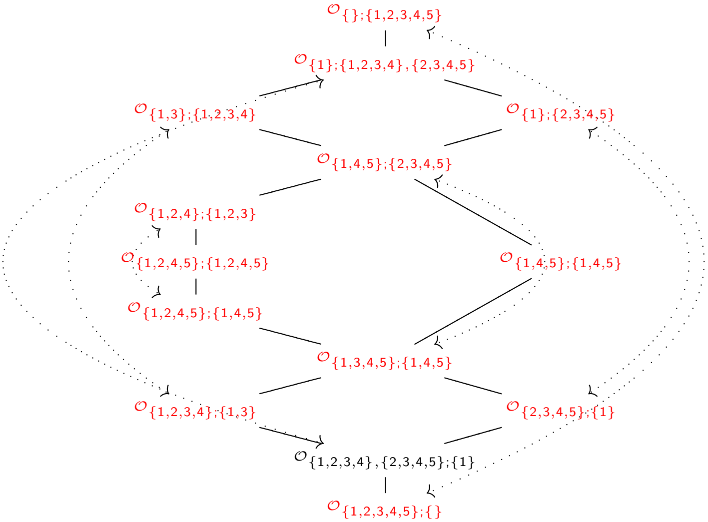
Example: Special Orbits of $D_5 \approx \mathfrak{so}(5,5)_{\mathbb{C}}$



Richardson Orbits of D_5



Tau Signatures of Special Orbits of D_5



Tau signatures for cells in the big block of $SO(5,5)$

- 365 representations with inf. char. ρ in big block
- 32 cells in the big block

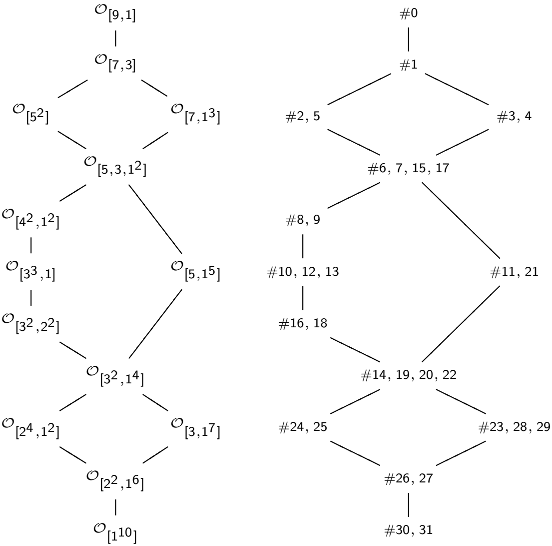
Output of `extract-cells`

```
// Individual cells.
// cell #0:
0[0]: {}

// cell #1:
0[1]: {2} --> 1,2
1[3]: {1} --> 0
2[5]: {3} --> 0,3,4
3[13]: {5} --> 2
4[14]: {4} --> 2
  *
  *
  *
// cell #29:
0[328]: {1,2,4,5} --> 2,3
1[340]: {2,3,4,5} --> 2
2[358]: {1,3,4,5} --> 0,1
3[364]: {1,2,3} --> 0

// cell #30:
0[353]: {1,2,3,4,5}

// cell #31:
0[357]: {1,2,3,4,5}
```

cell #	tau signature
0	{} , {1,2,3,4,5}
1	{1} , {1,2,3,4}
2	{1} , {2,3,4,5}
3	{1,3} , {1,3,4,5}
*	*
*	*
*	*
28	{2,3,4,5} , {1}
29	{2,3,4,5} , {1}
30	{1,2,3,4,5} , {}
31	{1,2,3,4,5} , {}

Each of these coincides with the tau signature of a particular nilpotent orbit.

Cell-Orbit Correspondences for $SO(5,5)$



More Generally:

Exceptional Groups: tables by Spaltenstein list induced orbits and Hasse diagrams.

Tau signatures of special orbits can be done by hand.

1. Use Spaltenstein's tables to figure out which special orbits are Richardson orbits and to identify the std Γ's corresponding to the corresponding Levi subalgebra.

2. Place the Richardson orbits in the Hasse diagram of special orbits, and then figure out the Γ parameters of the minimal Richardson orbits that contain a given special orbit and the minimal Richardson orbits that contain its Spaltenstein dual

Even E_8 can be done by hand.

Classical Groups:

Partition classification $\longrightarrow$ closure relations

Just need to

- which partitions correspond to special orbits (easy recipes in Collingwood-McGovern)
- use dominance ordering of partitions to partial order special orbits
- use formulas in [C-M] to determine partitions corresponding to Richardson orbits for each $\Gamma \in \Psi$. Place these in the Hasse diagram of special orbits and at the same time partial order Ψ.
- Use the partial ordering of Ψ to ascribe tau signatures to cells (employing `atlas` data)
- match orbit tau sigs to cell tau sigs

The Structure of $Prim(\mathfrak{g})_\lambda$

Standard modules and Irr HC-modules arise naturally in the study of $\widehat{G}_{\mathbb{R},adm}$

Verma and Irr HW modules much more convenient family for discussing primitive ideals.

Set

$\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$: Borel subalgebra of $\mathfrak{g}$ $\qquad \rho = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{h})}\alpha$

Theorem

Let $\lambda \in \mathfrak{h}^*$ and let $M(\lambda)$ denote the Verma module of highest weight $\lambda - \rho$; i.e., the left $U(\mathfrak{g})$-module

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$$

Then
(i) The Verma module $M(\lambda)$ has a unique irreducible quotient module $L(\lambda)$ which is of highest weight $\lambda - \rho$.
(ii) Every irreducible highest weight module is isomorphic to some $L(\lambda)$.

Duflo's theorem

Theorem (Duflo)

For $\lambda \in \mathfrak{h}^*$ *let*

$$L_\lambda = \textit{unique irreducible quotient of } M(\lambda)$$

Then

$$\varphi : W \to Prim\left(\mathfrak{g}\right)_\rho : w \to Ann\left(L_w\right)$$

is a surjection.

For $w \in W$ set

$$M_w = M_{-w\rho}$$

Parameterizing $Prim\left(\mathfrak{g}\right)_\rho$ is tantamount to understanding the fiber of $\varphi : W \to Prim\left(\mathfrak{g}\right)_\rho$

Definition

Let $\approx$ be the equivalence relation on W defined by

$$w \approx w' \iff \mathcal{O}_{Ann(L_w)} = \mathcal{O}_{Ann(L_{w'})}$$

The corresponding equivalence classes of elements of W are **double cells** in W.

Definition

Let $\sim$ be the equivalence relation on W defined by

$$w \sim w' \iff Ann(L_w) = Ann(L_{w'})$$

The corresponding equivalence classes of elements of W are called **left cells** in W.

Two pictures of $Prim(\mathfrak{g})_\rho$

HW-modules

$$\begin{array}{ccc}
W & \{L_w \mid w \in W\} & \text{same inf char} \\
\cup & \cup & \\
\mathcal{C} : dbl\ \ cell & \{L_w \mid w \in \mathcal{C}\} & \text{same nilpotent orbit} \\
\cup & \cup & \\
\ell : left\ \ cell & \{L_w \mid w \in \ell\} & \text{same primitive ideal}
\end{array}$$

HC-modules

$$\begin{array}{ccc}
B : \text{ block of HC-modules} & \{\pi_x \mid x \in B\} & \text{same inf char} \\
\cup & \cup & \\
C : \text{cell of HC-modules} & \{\pi_x \mid x \in C\} & \text{same nilpotent orbit} \\
\cup & \cup & \\
? & \{\pi_x \mid x \in ?\} & \text{same primitive ideal}
\end{array}$$

Connection with Weyl group reps

Let

$$p_w(\mu) := \text{Goldie rank of } U(\mathfrak{g})/Ann(L_{w\mu}) \qquad , \qquad \mu \in \Lambda^+$$

Theorem (Joseph)

p_w extends to a homogeneous harmonic polynomial on $\mathfrak{h}^*$ and $\sigma_w = \mathbb{C}\langle W \cdot p_w\rangle$ is an irreducible representation of W.

Theorem (Joseph)

Fix a finite-dimensional representation σ of W. The $w \in W$ such $\sigma_w \approx \sigma$ comprise the **double cell** *$\mathcal{C}_\sigma$ in W; i.e.*

$$\sigma_w \approx \sigma_{w'} \quad \Longrightarrow \quad Ann(L_w) \text{ and } Ann(L_{w'}) \text{ share the same nilpotent orbit}$$

Remark: The irr reps $\sigma \in \widehat{W}$ that arise in this fashion are exactly the **special representations** of W (when λ reg int)

Key results from theory of primitive ideals

$$\begin{tikzcd} U(\mathfrak{g}) \supset Ann(L_w) \arrow[rr, "gr"] \arrow[dr, "p_w"'] & & \mathcal{O}_w \subset \mathcal{N}_{\mathfrak{g}} \arrow[dl, "Springer"] \\ & \sigma_w \in \widehat{W} & \end{tikzcd}$$

Theorem (Barbasch-Vogan, Joseph)

Let $C \subset W$ be a double cell and let $\sigma \in \widehat{W}$ be the associated special W-rep. Then

$$\#\left\{ Ann(L_w) \mid w \in C \right\} = \dim \sigma$$

or, put another way,

$$\#\left\{ J \in Prim\left(\mathfrak{g}\right)_\rho \text{ sharing same orbit } \mathcal{O}_w \right\} = \dim \sigma_w$$

$L_w := L(-w\rho)$: simple HWM of highest weight $-w\rho - \rho$

$I_w := Ann\,(L_w)$

- I_{w_o} : unique max ideal (augmentation ideal, annihilator of triv rep)
- I_e = unique min PI at inf char ρ
- I_{s_α}, $\alpha \in \Pi$: "pen-minimal" ideals

Theorem

The primitive ideals I_{s_α}, $\alpha \in \Pi$, are all distinct from each other and I_e. Any primitive ideal strictly containing I_e contains at least one of the I_{s_α}.

Definition

The **tau invariant** of a primitive ideal I containing I_e is

$$\tau(I) = \{\alpha \in \Pi \mid I_{s_\alpha} \subseteq I\}$$

Theorem (Vogan)

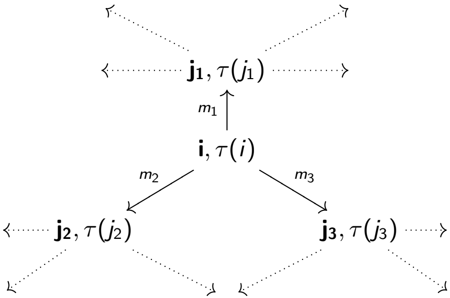
Let x be an element of a cell C of HC modules and let $\tau(x)$ be its descent set (from W-graph of C). Then

$$\tau(x) = \textit{tau-invariant of } Ann(x)$$

A partitioning of W-cells

Recall W-graph of cell: for each element $i \in C$ we attach

- a vertex $v[i]$
- a tau invariant $\tau[i] =$ tau invariant of $Ann(\pi_i)$
- a list of edges with multiplicities $e[i] = [(j_1, m_1), (j_2, m_d), \ldots, (j_k, m_k)]$



τ_0 subcells

Definition

We say two cell vertices x, y belong to the same τ_0-subcell and write

$$x \sim_{\tau_0} y$$

whenever

$$\tau(x) = \tau(y)$$

Obviously,

$$C = \coprod_{[x]_0 \in C/\sim_{\tau_0}} [x]_0$$

(Collecting together reps with common assoc variety and common τ-invariant)

A partitioning of cells, cont'd

τ_1 **subcells**: Set $\tau_1(x) = \{\tau(y) \mid x \to y \text{ is an edge}\}$

$$x \sim_{\tau_1} y \iff \tau(x) = \tau(y) \text{ and } \tau_1(x) = \tau_1(y)$$

$$C = \coprod_{[x]_1 \in C/\sim_{\tau_1}} [x]_1$$

τ_2 **subcells**: Set $\tau_2(x) = \{\tau_1(y) \mid x \to y \text{ is an edge}\}$

$$x \sim_{\tau_2} y \iff \tau_0(x) = \tau_0(y), \tau_1(x) = \tau_1(y), \tau_2(x) = \tau_2(y)$$

$$C = \coprod_{[x]_2 \in C/\sim_{\tau_2}} [x]_2$$

$\vdots$

τ_i **subcells**: Set $\tau_i(x) = \{\tau_{i-1}(y) \mid x \to y \text{ is an edge}\}$

$$x \sim_{\tau_i} y \iff \tau_0(x) = \tau_0(y), \ldots, \tau_i(x) = \tau_i(y) \text{ and}$$

$$C = \coprod_{[x]_i \in C/\sim_{\tau_i}} [x]_i$$

Def. τ_∞ **subcells** := final stable partitioning : $C = \coprod_{[x]_\infty \in C/\sim_\infty} [x]_{\tau_\infty}$

Lemma

The τ_∞ partitioning of a cell of HC-modules is compatible with the partitioning of the cell into subcells consisting of representations with the same primitive ideal:

$$Ann(x) = Ann(y) \quad \Longrightarrow \quad x \text{ and } y \text{ live in same } \tau_\infty\text{-subcell.}$$

(follows from well-definedness of Translation Functor for primitive ideals)

Conjecture (Vogan (1979))

Suppose J, J' are primitive ideals with the same infinitesimal character. Then

$$\tau_\infty(J) = \tau_\infty(J') \quad \Longleftrightarrow \quad J = J'$$

Theorem (Vogan, Garfinkle)

Vogan's Conjecture is true for the classical Lie algebras of type A_n , B_n and C_n

Theorem

Let C be any cell of irr adm reps with reg int inf char, in any real form of any exceptional group G. Then the τ_∞ partitioning of C coincides precisely with the partitioning of the cell into sets of irr HC modules sharing the same primitive ideal:

$$x \sim_\infty y \iff Ann(x) = Ann(y)$$

Proof:

- $\# P_\infty$ -subcells in C = dim special W-rep attached to C (explicit computation)
- Primitive ideal theory: max # prim ideals attached to $\mathcal{O}_C$ = dim special W-rep attached to C
- Compatibility of τ-partitioning scheme with partitioning by primitive ideals

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$F_4(\mathbb{R}) \times F_4(\mathbb{R})$ block

#	$\lvert C \rvert$	$\mathcal{O}_C$	σ_{spec}	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	F_4	$\phi_{1,0}$	1^1	$\phi_{1,0}$
1	6	$F_4(a_1)$	$\phi_{4,1}$	$1^2 2^2$	$\phi_{4,1} + \phi''_{2,4}$
3	6	$F_4(a_1)$	$\phi_{4,1}$	$1^2 2^2$	$\phi_{4,1} + \phi''_{2,4}$
4	9	$F_4(a_2)$	$\phi_{9,2}$	1^9	$\phi_{9,2}$
5	9	$F_4(a_2)$	$\phi_{9,2}$	1^9	$\phi_{9,2}$
11	9	$F_4(a_2)$	$\phi_{9,2}$	1^9	$\phi_{9,2}$
12	8	C_3	$\phi'_{8,3}$	1^8	$\phi'_{8,3}$
2	8	B_3	$\phi''_{8,3}$	1^8	$\phi''_{8,3}$
6	8	B_3	$\phi''_{8,3}$	1^8	$\phi''_{8,3}$
7	8	B_3	$\phi''_{8,3}$	1^8	$\phi''_{8,3}$
8	8	B_3	$\phi''_{8,3}$	1^8	$\phi''_{8,3}$
9	57	$F_4(a_3)$	$\phi_{12,4}$	$2^3 3^1 5^3 6^4 9^1$	$\phi_{12,4} + \phi_{16,5} + 2\phi''_{9,6} + \phi''_{6,6} + \phi''_{4,7} + \phi''_{1,12}$
13	47	$F_4(a_3)$	$\phi_{12,4}$	$5^4 4^4 3^3 2^1$	$\phi_{12,4} + \phi_{16,5} + \phi''_{9,6} + \phi'_{6,6} + \phi''_{4,7}$
14	72	$F_4(a_3)$	$\phi_{12,4}$	$4^6 6^2 9^4$	$\phi_{12,4} + 2\phi_{16,5} + \phi'_{9,6} + \phi''_{9,6} + \phi''_{6,6} + \phi_{4,8}$
18	8	$\tilde{A}_2$	$\phi'_{8,9}$	1^8	$\phi'_{8,9}$
10	8	A_2	$\phi''_{8,9}$	1^8	$\phi''_{8,9}$
15	8	A_2	$\phi''_{8,9}$	1^8	$\phi''_{8,9}$
16	8	A_2	$\phi''_{8,9}$	1^8	$\phi''_{8,9}$
21	8	A_2	$\phi''_{8,9}$	1^8	$\phi''_{8,9}$
17	9	$A_1 + \tilde{A}_1$	$\phi_{9,10}$	1^9	$\phi_{9,10}$
19	9	$A_1 + \tilde{A}_1$	$\phi_{9,10}$	1^9	$\phi_{9,10}$