Deficits in the U.S. Curriculum

1. Insufficient attention to place value

2. Insufficient attention to word problems.

3. Insufficient attention to units.

4. Insufficient attention to structure, reasoning, and justification.

5. Don't get me started on the number line.

1. Insufficient attention to place value

C. F. Gauss (according to Eaves):

The greatest calamity in the history of science
was the failure of Archimedes
to invent positional notation.

Four levels of sophistication:

$$432 = 400 + 30 + 2 = 4 \times 100 + 3 \times 10 + 2 = 4 \times 10^2 + 3 \times 10^1 + 2 \times 10^0$$

Teacher quotation:

I find that my teaching improved immensely,
not by learning advanced mathematics
but by going back and relearning the sophistication of place value and arithmetic.

In the US, place value is often/typically treated as a vocabulary item rather than the central organizing principle for arithmetic computation.

$$432 = 400 + 30 + 2 = 4 \times 100 + 3 \times 10 + 2 = 4 \times 10^2 + 3 \times 10^1 + 2 \times 10^0$$

JRME, May 2009: Of 15 teacher candidates interviewed,

3 (= 20%) thought of 432 as the third expression,

2 more (= 13 %) thought of it in terms of the second expression,

10 (= 2/3 = 67%) thought of it as the first expression (= concatenated digits, as in a telephone number).

Basic ideas:

1) every whole number is expressed as a sum of special numbers

(= a digit times a power of 10. There is no standard name!)

'very round number"? "single place number"?

http://www.maa.org/pmet/resources/PVHoweEpp-Nov2008.pdf

2) The arithmetic operations are performed by

a) operating with pairs of very round numbers; and

b) combining the results according to the rules of arithmetic.

First grade teaching sequence for beginning arithmetic:

US:

learn the facts (methods: memorization, doubles, doubles ± 1),

then progress to the algorithm (in recent times, with manipulatives).

A major publisher's draft text dealt with two-digit subtraction as a calculator topic.

East Asia:

learn the facts (addition and subtraction together) to 10 in a robust and flexible manner;

learn the teen numbers as a 10 and some 1s;

learn the higher addition/subtraction facts;

learn two-digit numbers as some 10s and some 1s;

learn two-digit addition/subtraction without regrouping
in terms of working with 10s and 1s independently

refine to deal with regrouping by relating back to the addition/subtraction facts;

proceed to the algorithm.

Note: develop mental math capacity.

Video lessons of Presidential Award candidates:

several had realized that place value is a key concept, and were attempting to put more emphasis on it,

but

i) they were working in isolation;

ii) none had come close to integrating it with arithmetic instruction a la East Asia.

Next step:

relative place value:

each place is 10 times as large as the place to its right

and (consequently) 1/10 as large as the place to the left.

gives rise to:

i) rationale for decimal fractions

ii) efficient approximation, including approximate addition and multiplication.

Take away: the first (leftmost) digit tells you most (usually, the dominant part) of the number.

Japanese Mathematics Standards:

i) grades 1 - 4: discuss relative sizes of numbers

ii) in grade 4: summarize the decimal system

I have never seen such standards in the US. (Common Core?)

Singapore:

with each new decimal place, the bundling process is carefully reviewed; and

the expanded form is recalled.

The expanded form is used in developing/examining algorithms.

2. Insufficient attention to word problems.

results in deficits in conceptions of the operations.

taxonomy of one-step addition/subtraction problems:

relatively few of the dozen types are systematically presented

Singapore 3rd grade:

36 students in a class, 4 more boys than girls. How many boys?

Singapore model method.

3. Insufficient attention to units.

What is the nature of number?
(Adjective or noun?)

Susan Lamon, Second Handbook of Research in Mathematics Education,
chapter on Rational Number and Proportional Thinking:

We do not know how to teach rational number.

Estimate: on a conservative definition of "proportional thinking",
about 10% of US adults do so.

Major US publisher:
second grade addition and subtraction word problems:

units suppressed in answers: either provided for the student, or just ignored.

comparison problems in which numbers of different kinds of things are compared,
obscuring the principle that "each number refers to a unit"

reciprocal relationship between size of unit and number of units in a given quantity

difficulties with addition of fractions:

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

is an error of inattention to units.

scaling: length vs. area vs. volume (3 square feet in a square yard)

Failure to define units in setting up variables.

NAEP Example: In a problem about small tables and large tables,
the solution rubric began:

Let s be small chairs.

At a certain college, there are 6 students for each professor.
If P = number of professors, and S = number of students,
write a relationship between P and S.

6S = P. (!!)

4. Insufficient attention to structure, reasoning, and justification

Arithmetic

lack of models for multiplication: array model, area model, stretching model

lack of discussion of uses of Rules of Arithmetic

FOIL vs. EWE

lack of systematic development of number line

Algebra

algebraic derivations as reasoning processes; the (missing) corona of reasons.

confusion about the nature and validity of derivations/proofs of identities;
introduction of spurious roots $\leftrightarrow$ non-reversible steps

example from recent Mathematics Teacher

Geometry

devaluation of proof in geometry courses

Functions

lack of conceptual development of fractional powers, and irrational powers -

lack of attention to monotonicity

lack of conceptual development of exponential function.

Laws of Exponents, rather than Law of Exponents and consequences.

failure to characterize the exponential function in terms of the Law of Exponents.

use of functions without adequate discussion of domain
(Mathematics Teacher, Dec 2010/Jan 2011)