Complex-analytic structures on moment-angle manifolds

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1. Moment-angle complexes and manifolds.

$\mathcal{K}$ an (abstract) simplicial complex on the set $[m] = \{1, \ldots, m\}$.

$I = \{i_1, \ldots, i_k\} \in \mathcal{K}$ a simplex. Always assume $\varnothing \in \mathcal{K}$.
Allow $\{i\} \notin \mathcal{K}$ for some i (ghost vertices).

Consider the unit polydisc in $\mathbb{C}^m$,

$$\mathbb{D}^m = \Big\{(z_1, \ldots, z_m) \in \mathbb{C}^m \colon |z_i| \leqslant 1, \quad i = 1, \ldots, m\Big\}.$$

Given $I \subset [m]$, set

$$B_I := \Big\{(z_1, \ldots, z_m) \in \mathbb{D}^m \colon |z_j| = 1 \text{ for } j \notin I\Big\}.$$

Following [**BP**] define the moment-angle complex

$$\mathcal{Z}_{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} B_I \subset \mathbb{D}^m$$

It is invariant under the coordinatewise action of the standard torus

$$\mathbb{T}^m = \Big\{(z_1, \ldots, z_m) \in \mathbb{C}^m \colon |z_i| = 1, \quad i = 1, \ldots, m\Big\}$$

on $\mathbb{C}^m$.

Constr 1 ($\mathcal{K}$-power)**.** Let X be a space, and W a subspace of X. Given $I \subset [m]$, set

$$(X, W)^I = \Big\{(x_1, \ldots, x_m) \in X^m \colon x_j \in A \text{ for } j \notin I\Big\} \cong \prod_{i \in I} X \times \prod_{i \notin I} W,$$

and define the $\mathcal{K}$-power (also known as the polyhedral product) of (X, W) as

$$(X, W)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (X, W)^I \subset X^m.$$

Then $\mathcal{Z}_{\mathcal{K}} = (\mathbb{D}, \mathbb{T})^{\mathcal{K}}$, where $\mathbb{T}$ is the unit circle.

Another important example is the complement of the coordinate subspace arrangement corresponding to $\mathcal{K}$:

$$U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1, \ldots, i_k\} \notin \mathcal{K}} \{z \in \mathbb{C}^m \colon z_{i_1} = \ldots = z_{i_k} = 0\},$$

namely,

$$U(\mathcal{K}) = (\mathbb{C}, \mathbb{C}^\times)^{\mathcal{K}},$$

where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

Clearly, $\mathcal{Z}_{\mathcal{K}} \subset U(\mathcal{K})$. Moreover, $\mathcal{Z}_{\mathcal{K}}$ is a $\mathbb{T}^m$-equivariant deformation retract of $U(\mathcal{K})$ for every $\mathcal{K}$ [**BP**, Th. 8.9].

Prop 1 ([**BP**]). *Assume* $|\mathcal{K}| \cong S^{n-1}$ *(a sphere triangulation with* m *vertices). Then* $\mathcal{Z}_K$ *is a closed manifold of dimension* $m+n$*.*

We refer to such $\mathcal{Z}_\mathcal{K}$ as moment-angle manifolds.

If $\mathcal{K} = \mathcal{K}_P$ is the dual triangulation of a simple convex polytope P, then $\mathcal{Z}_P = \mathcal{Z}_{\mathcal{K}_P}$ embeds in $\mathbb{C}^m$ as a nondegenerate (transverse) intersection of $m-n$ real quadratic hypersurfaces [**BM**], [**BP**]. Therefore, $\mathcal{Z}_P$ can be smoothed canonically.

Now we shall look at a wider class of simplicial complexes $\mathcal{K}$: starshaped spheres, or underlying complexes of complete simplicial fans.

A set of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k \in \mathbb{R}^n$ generates a convex polyhedral cone

$$\sigma = \{\mu_1 \mathbf{a}_1 + \ldots + \mu_k \mathbf{a}_k \colon \mu_i \in \mathbb{R},\ \mu_i \geqslant 0\}.$$

A cone is rational if its generators can be chosen from $\mathbb{Z}^n \subset \mathbb{R}^n$, and is strongly convex if it does not contain a line. A cone is simplicial (respectively, regular) if it is generated by a part of basis of $\mathbb{R}^n$ (respectively, $\mathbb{Z}^n$).

A fan is a finite collection $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ of strongly convex cones in $\mathbb{R}^n$ such that every face of a cone in Σ belongs to Σ and the intersection of any two cones in Σ is a face of each. A fan Σ is rational (respectively, simplicial, regular) if every cone in Σ is rational (respectively, simplicial, regular). A fan $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ is complete if $\sigma_1 \cup \ldots \cup \sigma_s = \mathbb{R}^n$.

Let Σ be a simplicial fan in $\mathbb{R}^n$ with m one-dimensional cones generated by $\mathbf{a}_1, \ldots, \mathbf{a}_m$. Its underlying simplicial complex is

$$\mathcal{K}_\Sigma = \bigl\{ I \subset [m] \colon \{\mathbf{a}_i \colon i \in I\} \text{ spans a cone of } \Sigma \bigr\}$$

Note: Σ is complete iff $|\mathcal{K}_\Sigma|$ is a triangulation of S^{n-1}.

Now consider the linear map

$$\Lambda_{\mathbb{R}} \colon \mathbb{R}^m \to \mathbb{R}^n, \quad \mathbf{e}_i \mapsto \mathbf{a}_i,$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_m$ is the standard basis of $\mathbb{R}^m$. Set

$$\mathbb{R}^m_> = \{(y_1, \ldots, y_m) \in \mathbb{R}^m \colon y_i > 0\},$$

and define

$$R_\Sigma := \exp(\operatorname{Ker} \Lambda_{\mathbb{R}}) = \Big\{(y_1, \ldots, y_m) \in \mathbb{R}^m_> \colon \prod_{i=1}^m y_i^{\langle \mathbf{a}_i, \mathbf{u} \rangle} = 1 \text{ for all } \mathbf{u} \in \mathbb{R}^n\Big\},$$

Note: $R_\Sigma \cong \mathbb{R}^{m-n}_>$ if Σ is complete (or contains an n-dimensional cone).

Both $\mathbb{R}^m_>$ and its subgroup R_Σ act on the complement $U(\mathcal{K}_\Sigma) \subset \mathbb{C}^m$ by coordinatewise multiplications.

Thm 1. *Let Σ be a complete simplicial fan in $\mathbb{R}^n$ with m one-dimensional cones, and let $\mathcal{K} = \mathcal{K}_\Sigma$ be its underlying simplicial complex. Then*

(a) the group R_Σ acts on $U(\mathcal{K})$ freely and properly, and the quotient $U(\mathcal{K})/R_\Sigma$ has a canonical structure of a smooth $(m+n)$-dimensional manifold;

(b) $U(\mathcal{K})/R_\Sigma$ is $\mathbb{T}^m$-equivariantly homeomorphic to $\mathcal{Z}_\mathcal{K}$.

Therefore, $\mathcal{Z}_\mathcal{K}$ can be smoothed canonically.

Rem 1. The construction of the smooth structure on $\mathcal{Z}_{\mathcal{K}_\Sigma}$ from Thm 1 *does* depend on the geometry of the fan Σ.

However, we expect that the smooth structures coming from fans Σ and Σ' are equivalent whenever the underlying simplicial complexes $\mathcal{K}_\Sigma$ and $\mathcal{K}_{\Sigma'}$ are the same. This question is equivalent to that the quotients $\mathcal{Z}_{\mathcal{K}_\Sigma}/\mathbb{T}^m$ and $\mathcal{Z}_{\mathcal{K}_{\Sigma'}}/\mathbb{T}^m$ are diffeomorphic as manifolds with corners whenever $\mathcal{K}_\Sigma = \mathcal{K}_{\Sigma'}$. It is true in the polytopal case, and also for those fans Σ which are shellable.

Question 1. *Describe the class of sphere triangulations $\mathcal{K}$ for which the moment-angle manifold $\mathcal{Z}_{\mathcal{K}}$ admits a smooth structure.*

2. Complex-analytic structures.

We shall show that the even-dimensional moment-angle manifold $\mathcal{Z}_\mathcal{K}$ corresponding to a complete simplicial fan admits a structure of a complex manifold. The idea is to replace the action of $\mathbb{R}^{m-n}_>$ on $U(\mathcal{K})$ (whose quotient is $\mathcal{Z}_\mathcal{K}$) by a holomorphic action of $\mathbb{C}^{\frac{m-n}{2}}$ on the same space.

Rem 2. Complex structures on *polytopal* moment-angle manifolds $\mathcal{Z}_P$ were described by Bosio and Meersseman. Existence of complex structure on moment-angle manifolds corresponding to complete simplicial fans has been also recently and independently established by Tambour.

Assume $m-n$ is even from now on. We can always achieve this by formally adding an 'empty' one-dimensional cone to Σ; this corresponds to adding a ghost vertex to $\mathcal{K}$, or multiplying $\mathcal{Z}_\mathcal{K}$ by a circle. The column of matrix $\Lambda_\mathbb{R}$ corresponding to the 'empty' 1-cone is set to be zero.

Set $\ell = \frac{m-n}{2}$.

Constr 2. Choose a linear map $\Psi\colon \mathbb{C}^\ell \to \mathbb{C}^m$ satisfying two conditions:

(a) $\mathrm{Re}\circ\Psi\colon \mathbb{C}^\ell \to \mathbb{R}^m$ is a monomorphism.

(b) $\Lambda_{\mathbb{R}}\circ\mathrm{Re}\circ\Psi = 0$.

This corresponds to choosing a complex structure and specifying a complex basis in the real vector space $\mathrm{Ker}\,\Lambda_{\mathbb{R}} \cong \mathbb{R}^{2\ell}$. We also obtain that the composite map of the top line in the following diagram is zero:

$$\begin{array}{ccccccc}
\mathbb{C}^\ell & \xrightarrow{\Psi} & \mathbb{C}^m & \xrightarrow{\mathrm{Re}} & \mathbb{R}^m & \xrightarrow{\Lambda_{\mathbb{R}}} & \mathbb{R}^n \\
 & & \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} \\
 & & (\mathbb{C}^\times)^m & \xrightarrow{|\cdot|} & \mathbb{R}^m_> & \xrightarrow{\exp\Lambda_{\mathbb{R}}} & \mathbb{R}^n_>
\end{array}$$

where $|\cdot|$ denotes the map $(z_1,\ldots,z_m)\mapsto(|z_1|,\ldots,|z_m|)$. Now set

$$C_{\Psi,\Sigma} = \exp\Psi(\mathbb{C}^\ell) = \Bigl\{\Bigl(e^{\langle\psi_1,\mathbf{w}\rangle},\ldots,e^{\langle\psi_m,\mathbf{w}\rangle}\Bigr)\in(\mathbb{C}^\times)^m\Bigr\}$$

where $\mathbf{w}=(w_1,\ldots,w_\ell)\in\mathbb{C}^\ell$, ψ_i denotes the ith row of the $m\times\ell$-matrix $\Psi=(\psi_{ij})$.

Then $C_{\Psi,\Sigma}\cong\mathbb{C}^\ell$ is a complex-analytic (but not algebraic) subgroup in $(\mathbb{C}^\times)^m$. It acts on $U(\mathcal{K})$ by holomorphic transformations.

Ex 1. Let $\mathcal{K}$ be empty on 2 elements (that is, $\mathcal{K}$ has two ghost vertices). We therefore have $n = 0$, $m = 2$, $\ell = 1$, and $\Lambda_{\mathbb{R}}\colon \mathbb{R}^2 \to 0$ is a zero map. Let $\Psi\colon \mathbb{C} \to \mathbb{C}^2$ be given by $z \mapsto (z, \alpha z)$ for some $\alpha \in \mathbb{C}$, so that

$$C = C_{\Psi,\Sigma} = \big\{(e^z, e^{\alpha z})\big\} \subset (\mathbb{C}^\times)^2.$$

Condition (b) of Constr 2 is void, while (a) is equivalent to that $\alpha \notin \mathbb{R}$. Then $\exp\Psi\colon \mathbb{C} \to (\mathbb{C}^\times)^2$ is an embedding, and the quotient $(\mathbb{C}^\times)^2/C$ with the natural complex structure is a complex torus $T^2_{\mathbb{C}}$ with parameter $\alpha \in \mathbb{C}$:

$$(\mathbb{C}^\times)^2/C \cong \mathbb{C}/(\mathbb{Z} \oplus \alpha\mathbb{Z}) = T^2_{\mathbb{C}}(\alpha).$$

Similarly, if $\mathcal{K}$ is empty on 2ℓ elements (so that $n = 0$, $m = 2\ell$), we may obtain any complex torus $T^{2\ell}_{\mathbb{C}}$ as the quotient $(\mathbb{C}^\times)^{2\ell}/C_{\Psi,\Sigma}$ [Meersseman].

Thm 2. *Let Σ be a complete simplicial fan in $\mathbb{R}^n$ with m one-dimensional cones, and let $\mathcal{K} = \mathcal{K}_\Sigma$ be its underlying simplicial complex. Assume that $m - n = 2\ell$. Then*

(a) *the holomorphic action of the group $C_{\Psi,\Sigma}$ on $U(\mathcal{K})$ is free and proper, and the quotient $U(\mathcal{K})/C_{\Psi,\Sigma}$ has a canonical structure of a compact complex manifold of complex dimension $m - \ell$;*

(b) *there is a $\mathbb{T}^m$-equivariant diffeomorphism $U(\mathcal{K})/C_{\Psi,\Sigma} \cong \mathcal{Z}_{\mathcal{K}}$ defining a complex structure on $\mathcal{Z}_{\mathcal{K}}$ in which $\mathbb{T}^m$ acts holomorphically.*

Rem 3. Unlike the smooth structure, the complex structure on $\mathcal{Z}_\mathcal{K}$ depends on both the geometry of Σ and the choice of Ψ. (The latter is already clear from the torus example (Ex 1).

Ex 2 (Hopf manifold)**.** Let Σ be the complete fan in $\mathbb{R}^n$ whose cones are generated by all proper subsets of $n+1$ vectors $\mathbf{e}_1,\ldots,\mathbf{e}_n,-\mathbf{e}_1-\ldots-\mathbf{e}_n$.

To make $m-n$ even we add one 'empty' 1-cone. We have $m=n+2$, $\ell=1$. Then $\Lambda_\mathbb{R}\colon\mathbb{R}^{n+2}\to\mathbb{R}^n$ is given by the matrix $(\mathbf{0}\ I\ -\mathbf{1})$, where I is the unit $n\times n$ matrix, and $\mathbf{0}$, $\mathbf{1}$ are the n-columns of zeros and units respectively.

We have that $\mathcal{K}$ is the boundary of an n-dim simplex with $n+1$ vertices and 1 ghost vertex, $\mathcal{Z}_\mathcal{K}\cong S^1\times S^{2n+1}$, and $U(\mathcal{K})=\mathbb{C}^\times\times(\mathbb{C}^{n+1}\setminus\{0\})$.

Take $\Psi\colon\mathbb{C}\to\mathbb{C}^{n+2}$, $z\mapsto(z,\alpha z,\ldots,\alpha z)$ for some $\alpha\in\mathbb{C}$, $\alpha\notin\mathbb{R}$. Then

$$C=C_{\Psi,\Sigma}=\bigl\{(e^z,e^{\alpha z},\ldots,e^{\alpha z})\colon z\in\mathbb{C}\bigr\}\subset(\mathbb{C}^\times)^{n+2},$$

and $\mathcal{Z}_\mathcal{K}$ acquires a complex structure as the quotient $U(\mathcal{K})/C$:

$$\mathbb{C}^\times\times\bigl(\mathbb{C}^{n+1}\setminus\{0\}\bigr)\big/\,\{(t,\mathbf{w})\sim(e^zt,e^{\alpha z}\mathbf{w})\}\cong\bigl(\mathbb{C}^{n+1}\setminus\{0\}\bigr)\big/\,\{\mathbf{w}\sim e^{2\pi i\alpha}\mathbf{w}\},$$

where $t\in\mathbb{C}^\times$, $\mathbf{w}\in\mathbb{C}^{n+1}\setminus\{0\}$. The latter quotient of $\mathbb{C}^{n+1}\setminus\{0\}$ is known as the Hopf manifold.

3. Holomorphic bundles over toric varieties and Hodge numbers.

Manifolds $\mathcal{Z}_\mathcal{K}$ corresponding to complete *regular* simplicial fans are total spaces of holomorphic principal bundles over toric varieties with fibre a complex torus, by a generalisation of the construction of Meersseman and Verjovsky. This allows us to calculate invariants of complex structures on $\mathcal{Z}_\mathcal{K}$.

A toric variety is a normal algebraic variety X on which an algebraic torus $(\mathbb{C}^\times)^n$ acts with a dense orbit.

Toric varieties are classified by rational fans. Under this correspondence,

$$\begin{aligned}
\text{complete fans} &\longleftrightarrow \text{compact varieties}\\
\text{normal fans of polytopes} &\longleftrightarrow \text{projective varieties}\\
\text{regular fans} &\longleftrightarrow \text{nonsingular varieties}\\
\text{simplicial fans} &\longleftrightarrow \text{orbifolds}
\end{aligned}$$

Σ complete, simplicial, rational;
$\mathbf{a}_1, \ldots, \mathbf{a}_m$ primitive integral generators of 1-cones.

Constr 3 ('Cox construction'). Let $\Lambda_{\mathbb{C}} \colon \mathbb{C}^m \to \mathbb{C}^n$, $\mathbf{e}_i \mapsto \mathbf{a}_i$,

$$\exp \Lambda_{\mathbb{C}} \colon (\mathbb{C}^\times)^m \to (\mathbb{C}^\times)^n,$$
$$(z_1, \ldots, z_m) \mapsto \Big(\prod_{i=1}^m z_i^{a_{i1}}, \ldots, \prod_{i=1}^m z_i^{a_{in}} \Big)$$

Set $G_\Sigma = \operatorname{Ker} \exp \Lambda_{\mathbb{C}}$.
This is an $(m-n)$-dimensional algebraic subgroup in $(\mathbb{C}^\times)^m$.
It acts almost freely (with finite isotropy subgroups) on $U(\mathcal{K}_\Sigma)$.
If Σ is regular, then $G_\Sigma \cong (\mathbb{C}^\times)^{m-n}$ and the action is free.

$X_\Sigma = U(\mathcal{K}_\Sigma)/G_\Sigma$ the toric variety associated to Σ.
The quotient torus $(\mathbb{C}^\times)^m/G_\Sigma \cong (\mathbb{C}^\times)^n$ acts on X_Σ with a dense orbit.

Observe that $C_{\Psi,\Sigma} \subset G_\Sigma$ as a complex ℓ-dimensional subgroup.

Prop 2.

(a) *The toric variety X_Σ is homeomorphic to the quotient of $\mathcal{Z}_{\mathcal{K}_\Sigma}$ by the holomorphic action of $G_\Sigma/C_{\Psi,\Sigma}$.*

(b) *If Σ is regular, then there is a holomorphic principal bundle $\mathcal{Z}_{\mathcal{K}_\Sigma} \to X_\Sigma$ with fibre the compact complex torus $G_\Sigma/C_{\Psi,\Sigma}$ of dimension ℓ.*

Rem 4. For singular varieties X_Σ the quotient projection $\mathcal{Z}_{\mathcal{K}_\Sigma} \to X_\Sigma$ is a holomorphic principal Seifert bundle for an appropriate orbifold structure on X_Σ (same as in the projective case of [Meersseman–Verjovsky]).

Given a complex n-dimensional manifold M, there is a decomposition $\Omega^*_{\mathbb{C}}(M) = \bigoplus \Omega^{p,q}(M)$ of the space of differential $\mathbb{C}$-forms on M into a direct sum of the subspaces of (p,q)-forms for $1 \leqslant p,q \leqslant n$, and the Dolbeault differential $\bar{\partial}\colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$.
$h^{p,q}(M) = \dim H^{p,q}_{\bar{\partial}}(M)$: the Hodge numbers of M.

The Dolbeault cohomology of a complex torus is given by

$$H^{*,*}_{\bar{\partial}}(T^{2\ell}_{\mathbb{C}}) \cong \Lambda[\xi_1,\ldots,\xi_\ell,\eta_1,\ldots,\eta_\ell],$$

where $\xi_1,\ldots,\xi_\ell \in H^{1,0}_{\bar{\partial}}(T^{2\ell}_{\mathbb{C}})$, $\eta_1,\ldots,\eta_\ell \in H^{0,1}_{\bar{\partial}}(T^{2\ell}_{\mathbb{C}})$. Hence, $h^{p,q}(T^{2\ell}_{\mathbb{C}}) = \binom{\ell}{p}\binom{\ell}{q}$.

The Dolbeault cohomology of a complete nonsingular toric variety X_Σ is given by [Danilov–Jurkiewicz]:

$$H^{*,*}_{\bar{\partial}}(X_\Sigma) \cong \mathbb{C}[v_1,\ldots,v_m]/(\mathcal{I}_{\mathcal{K}_\Sigma} + \mathcal{J}_\Sigma),$$

where $v_i \in H^{1,1}_{\bar{\partial}}(X_\Sigma)$,
$\mathcal{I}_{\mathcal{K}_\Sigma} = \bigl(v_{i_1}\cdots v_{i_k}\colon \{i_1,\ldots,i_k\} \notin \mathcal{K}_\Sigma\bigr)$ (the Stanley–Reisner ideal),
$\mathcal{J}_\Sigma = (\sum_{k=1}^m a_{kj}v_k, \quad 1 \leqslant j \leqslant n)$.
We have $h^{p,p}(X_\Sigma) = h_p$, where $(h_0,h_1,\ldots,h_n)$ is the h-vector of $\mathcal{K}_\Sigma$, and $h^{p,q}(X_\Sigma) = 0$ for $p \neq q$.

By an application of the Borel spectral sequence to the holomorphic bundle $\mathcal{Z}_{\mathcal{K}} \to X_\Sigma$ we obtain the following description of the Dolbeault cohomology.

Thm 3. *Let Σ be a complete rational nonsingular fan. Then the Dolbeault cohomology group $H^{p,q}_{\bar\partial}(\mathcal{Z}_{\mathcal{K}})$ is isomorphic to the (p,q)-th cohomology group of the differential bigraded algebra*

$$\Bigl[\Lambda[\xi_1,\ldots,\xi_\ell,\eta_1,\ldots,\eta_\ell]\otimes H^{*,*}_{\bar\partial}(X_\Sigma),d\Bigr]$$

whose differential d of bidegree $(0,1)$ is defined on the generators as

$$dv_i = d\eta_j = 0, \quad d\xi_j = c(\xi_j), \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant \ell,$$

where $c\colon H^{1,0}_{\bar\partial}(T^{2\ell}_{\mathbb{C}}) \to H^2(X_\Sigma,\mathbb{C}) = H^{1,1}_{\bar\partial}(X_\Sigma)$ is the first Chern class map of the torus principal bundle $\mathcal{Z}_{\mathcal{K}} \to X_\Sigma$.

This result may be compared to the analogous description of the ordinary cohomology of $\mathcal{Z}_{\mathcal{K}}$ from [**BP**]:

Thm 4. *$H^*(\mathcal{Z}_{\mathcal{K}})$ is isomorphic to the cohomology of the dga*

$$\Bigl[\Lambda[u_1,\ldots,u_{m-n}]\otimes H^*(X_\Sigma),d\Bigr],$$

with $\deg u_j = 1$, $\deg v_i = 2$, *and differential d defined on the generators as*

$$dv_i = 0, \quad du_j = \gamma_{j1}v_1 + \ldots + \gamma_{jm}v_m, \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant m-n.$$

Thm 5. *Let $\mathcal{Z}_\mathcal{K}$ be as in Thm 3, and let k be the number of ghost vertices in $\mathcal{K}$. Then the Hodge numbers $h^{p,q} = h^{p,q}(\mathcal{Z}_\mathcal{K})$ satisfy*

(a) $\binom{k-\ell}{p} \leqslant h^{p,0} \leqslant \binom{[k/2]}{p}$ *for* $p \geqslant 0$;

(b) $h^{0,q} = \binom{\ell}{q}$ *for* $q \geqslant 0$;

(c) $h^{1,q} = (\ell - k)\binom{\ell}{q-1} + h^{1,0}\binom{\ell+1}{q}$ *for* $q \geqslant 1$;

(d) $\frac{\ell(3\ell+1)}{2} - h_2(\mathcal{K}) - \ell k + (\ell+1)h^{2,0} \leqslant h^{2,1} \leqslant \frac{\ell(3\ell+1)}{2} - \ell k + (\ell+1)h^{2,0}$.

Rem 5. At most one ghost vertex is required to make $\dim \mathcal{Z}_\mathcal{K} = m + n$ even. Note that $k \leqslant 1$ implies $h^{p,0}(\mathcal{Z}_\mathcal{K}) = 0$, so that $\mathcal{Z}_\mathcal{K}$ does not have holomorphic forms of any degree in this case.

If $\mathcal{Z}_\mathcal{K}$ is a torus, then $m = k = 2\ell$, and $h^{1,0}(\mathcal{Z}_\mathcal{K}) = h^{0,1}(\mathcal{Z}_\mathcal{K}) = \ell$. Otherwise Thm 5 implies that $h^{1,0}(\mathcal{Z}_\mathcal{K}) < h^{0,1}(\mathcal{Z}_\mathcal{K})$, and therefore $\mathcal{Z}_\mathcal{K}$ is not Kähler (this was observed by [Meersseman] in the polytopal case).

Ex 3 (Calabi–Eckmann manifold). Let $X_\Sigma = \mathbb{C}P^p \times \mathbb{C}P^q$ with $p \leqslant q$, so $n = p+q$, $m = n+2$ and $\ell = 1$. The cohomology ring is $\mathbb{C}[x,y]/(x^{p+1}, y^{q+1})$. Choose $\Psi = (1, \ldots, 1, \alpha, \ldots, \alpha)^t$ where the number of units is $p+1$ and $\alpha \notin \mathbb{R}$. This provides $\mathcal{Z}_\mathcal{K} \cong S^{2p+1} \times S^{2q+1}$ with a structure of a complex manifold. It is the total space of a holomorphic principal bundle over $\mathbb{C}P^p \times \mathbb{C}P^q$ with fibre a complex torus $\mathbb{C}/(\mathbb{Z} \oplus \alpha\mathbb{Z})$, a Calabi–Eckmann manifold $\mathit{CE}(p,q)$.

By Thm 3, $\quad H^{*,*}_{\bar\partial}\bigl(\mathit{CE}(p,q)\bigr) \cong H\Bigl[\Lambda[\xi,\eta] \otimes \mathbb{C}[x,y]/(x^{p+1}, y^{q+1}), d\Bigr]$,
where $dx = dy = d\eta = 0$ and $d\xi = x - y$ for an appropriate choice of x, y. We therefore obtain

$$H^{*,*}_{\bar\partial}\bigl(\mathit{CE}(p,q)\bigr) \cong \Lambda[\omega,\eta] \otimes \mathbb{C}[x]/(x^{p+1}),$$

where $\omega \in H^{q+1,q}_{\bar\partial}\bigl(\mathit{CE}(p,q)\bigr)$ is the cohomology class of the cocycle $\xi\frac{x^{q+1}-y^{q+1}}{x-y}$. This calculation is originally due to Borel.

Ex 4. The product $S^3 \times S^3 \times S^5 \times S^5$ has has two complex structures as a product of Calabi–Eckmann manifolds, namely, $\mathit{CE}(1,1) \times \mathit{CE}(2,2)$ and $\mathit{CE}(1,2) \times \mathit{CE}(1,2)$.

In the first case $h^{2,1} = 1$, and $h^{2,1} = 0$ in the second.

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