Intersections of Quadrics and the Polyhedral Product Functor.

Santiago López de Medrano

A Personal View.
Banff, November 8, 2010.

Version 2, with some comments added
(which were expressed verbally during the talk).

- In 1978 C. Camacho, N. Kuiper and J. Palis published their work on the topological classification of holomorphic complex flows.
- A crucial example was the hyperbolic linear complex system of differential equations:

$$\dot{z}_i = \lambda_i z_i \quad i = 1, \ldots n$$

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This system has an immediate solution:

$$\dot{z}_i = \lambda_i z_i \quad i = 1, \ldots n$$

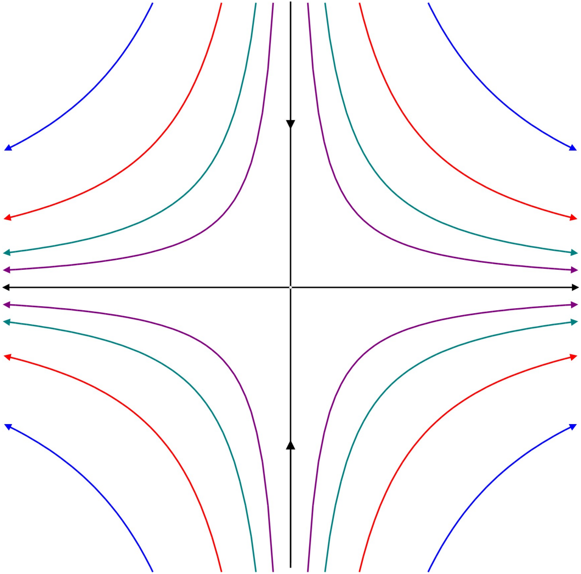
$$z_{i\tau} = z_{i_0} e^{\lambda_i \tau} \quad i = 1, \ldots n$$

(where τ is a complex parameter).

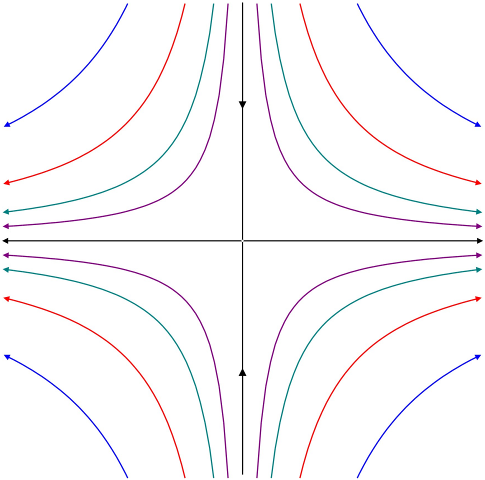
$$z_{i\tau} = z_{i_0} e^{\lambda_i \tau} \quad i = 1, \dots n$$

But behind this simple solution a great deal of geometry is hidden.

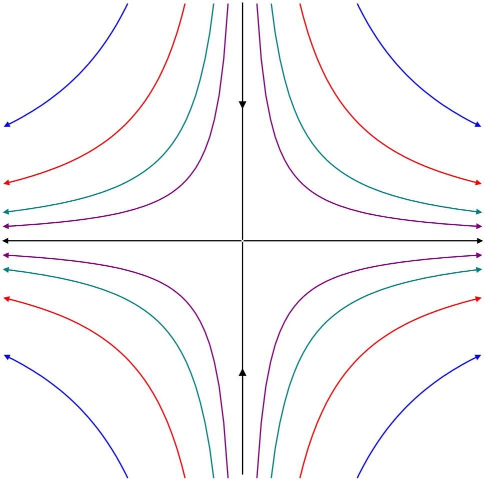
The solutions are real surfaces (complex 1-dimensional), usually called leaves.
We cannot draw the complex system. Here is a real analog that gives a partial idea:



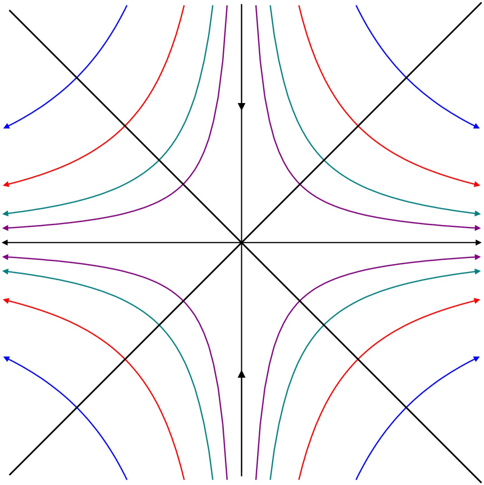
$$\dot{x} = x \qquad \dot{y} = -y$$



Leaves that approach the origin are called *Poincaré leaves* (in black) and those that keep their distance from the origin are called *Siegel leaves* (in color).



Poincaré leaves are responsible for the non-Haussdorff behavior of the system: observe how a sequence of Siegel leaves converges to two different Poincar'e leaves. In the complex case they also converge spiraling to the Poincaré leaves. This gives rise to moduli for the homeomorphism classification.



In the complement U of the *arrangement* of Poincaré subspaces the geometry is nice: in each Siegel leave there is a unique point closest to the origin. The set of such points is a transversal: it is the quotient of U by the action of $\mathbb{C}$ whose orbits are the leaves, but it lies inside U.

This means that the space of Siegel leaves is Hausdorff and in fact it is a variety: it is the set of non-zero $z \in \mathbb{C}^n$ that satisfy the following equation:

$$\Sigma \lambda_i \left| z_i \right|^2 = 0$$

This transversal is radial; it is enough to consider its intersection Z with the unit sphere.

$$\Sigma \lambda_i \left|z_i\right|^2 = 0$$
$$\Sigma \left|z_i\right|^2 = 1$$

- Questions:

  When is Z a *smooth* variety?

  What are all the possible *cases*?

  What is the topological type of Z?

- In January 1984 Alberto Verjovsky organized a seminar on Dynamical Systems at the Institute of Mathematics, UNAM, where he gave several talks about the Camacho-Kuiper-Palis results and raised the above questions.

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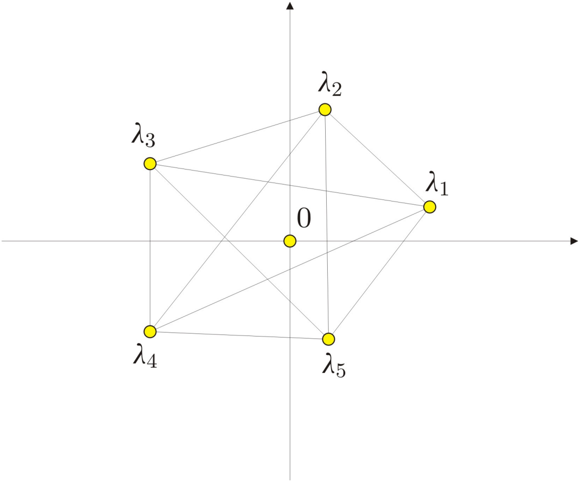
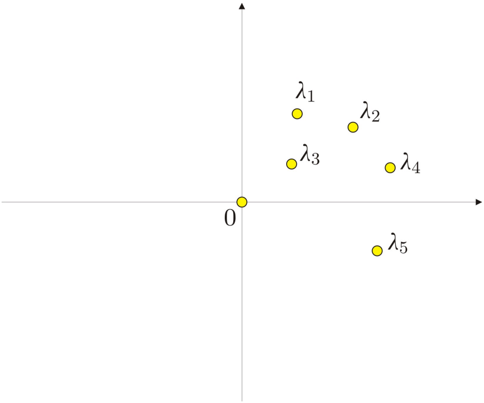
Everything depends on the position of the origin with respect to the coefficients λ_i.





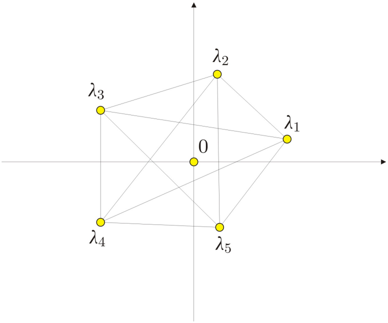
If they are all on one side of the origin Z is empty.

$$\Sigma\lambda_i\,|z_i|^2 = 0$$

$$\Sigma\,|z_i|^2 = 1$$

- Z is smooth, provided Λ satisfies the *weak hyperbolicity condition*:

  The origin is not in the segment joining two λ_i.



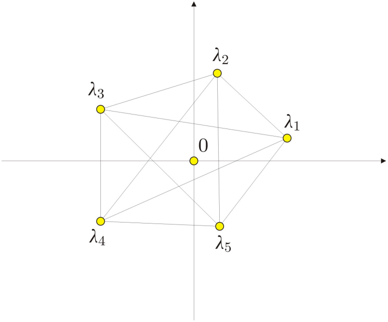
This has some immediate but very important consequences:

1 One can have two or more *equal* eigenvalues.

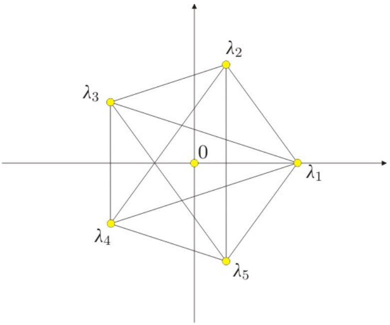
2 This gives many examples: take a regular polygon with an odd number of sides and take the $i-th$ vertex with multiplicity n_i.

3 The smooth type of Z does not change if we deform the coefficients as long as we do not violate the *weak hyperbolicity condition*.

First result: Every configuration can be deformed as above into a polygonal form with multiplicities. So any regular type can be specified by an odd cyclic partition n: $n = n_1 + n_2 + \cdots + n_m$, define up to cyclic permutation of the n_i.



λ_2
λ_3
λ_1
0
λ_4
λ_5



λ_2
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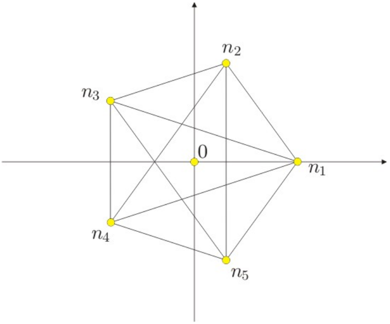

n_2
n_3
0
n_1
n_4
n_5

But now, how to discover the topological type of Z for each partition $n = n_1 + n_2 + \cdots + n_k$?

For this I observed that Z has a natural action of the n-torus $T^n = S^1 \times S^1 \cdots \times S^1$.

$$\Sigma \lambda_i |z_i|^2 = 0$$

$$\Sigma |z_i|^2 = 1$$

Just multiply every coordinate by a unit complex number.

The second observation was that the quotient of this action had a simple description:

Every point z is equivalent under this action to a unique point all whose coordinates are real and non-negative.

This is a *transversal* to the action: The quotient P can be realized as a subset of Z.

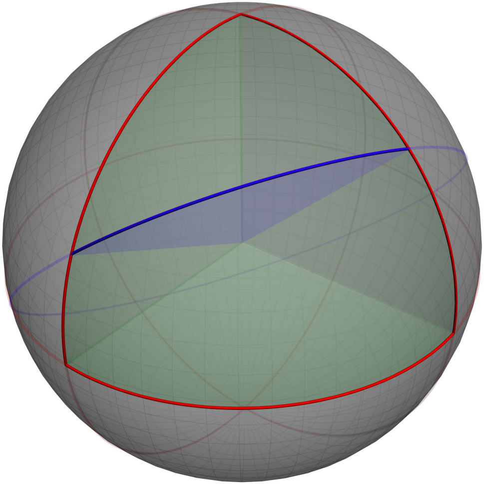
$$\Sigma\lambda_i |x_i|^2 = 0$$

$$\Sigma |x_i|^2 = 1$$

$$x_i \geq 0$$

So Z can be recovered from $P \times T^n$ just by identifying points according to the stabilizer corresponding to each point of P.

P is some sort of spherical polytope



which can be straightened by passing to the coordinates $r_i = x_i^2 (= |z_i|^2)$.

$$\Sigma\lambda_i \left|x_i\right|^2 = 0$$

$$\Sigma \left|x_i\right|^2 = 1$$

$$x_i \geq 0$$

$$\Sigma\lambda_i r_i = 0$$

$$\Sigma r_i = 1$$

$$r_i \geq 0$$

P can be seen as a convex polytope. The facets of P are given by the condition $r_i = 0$ (if not empty) which corresponds to the points of Z with $z_i = 0$, that is points where the i-th factor of the torus acts trivially.

So all the topology of Z is determined by either:

1. The odd cyclic partition $n = n_1 + n_2 + \cdots + n_m$ or, equivalently,
2. A simple convex polytope of dimension d with at most $d + 3$ facets, where $d = n - 3$.

The combinatorial properties of P can be deduced from 1: its faces are the non-empty intersections $Z \cap \{z_i = 0, i \in J\}$ where $J \subset \{1, \ldots, n\}$.

But we know when such intersection is not empty: when the rest of the λ_i have the origin in their convex hull.

In the inverse direction, it is easy to put any such P in the above form.

But, how make these combinatorial data spill out the information they hide?

One first idea is to find somewhere else a torus action with section on some manifold with quotient P.

This works out well for the triangular case $n = p + q + r$: By working out the combinatorics it follows that P is the product of 3 simplices $P = \Delta^{p-1} \times \Delta^{q-1} \times \Delta^{r-1}$ and it is clear that the standard action of T^n on the sphere S^{2n-1} has Δ^{n-1} as its quotient:

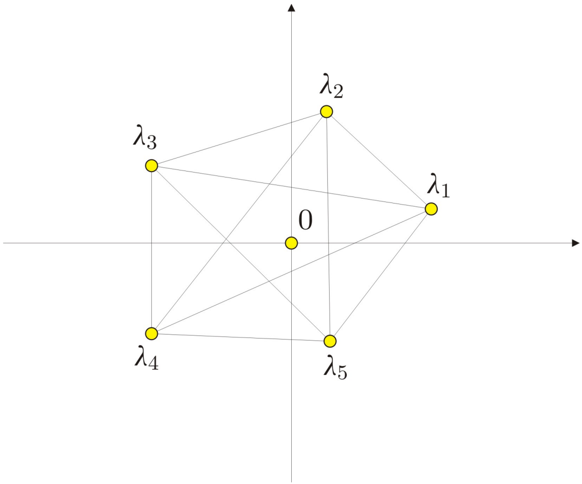
$$\Sigma \left|z_i\right|^2 = 1$$

$$\Sigma r_i = 1, \;\; r_i \geq 0.$$

Therefore, in this case Z is $S^{2p-1} \times S^{2q-1} \times S^{2r-1}$.

The next case would be the pentagonal one:
$n = n_1 + n_2 + n_3 + n_4 + n_5$ and take the simplest possibility:

$$5 = 1 + 1 + 1 + 1 + 1$$



WANTED:

Compact manifold of dimension 7.

It is simply connected.

Particular characteristic: it has an action of T^5 with a pentagon as transversal.

SURPRISE!

In a list by Dennis McGavran appears such a manifold:

$$Z = (S^3 \times S^4)\#(S^3 \times S^4)\#(S^3 \times S^4)\#(S^3 \times S^4)\#(S^3 \times S^4)$$

Unfortunately the method stops here.

There are not enough lists of torus actions in which to search for candidates: not a single additional example was found.

Nevertheless, MacGavran's example was a decisive clue as to what to expect.

How does one *prove* that a simply connected 7-manifold is such a connected sum?

I understood that I had to use the experience of the theory of *surgery of manifolds*, which says that in cases like this one should:

A.- Check the homotopy data: in this case the homology groups.

B.- Check the tangential data.

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B.- Check the tangential data.

Part B is trivial, our manifolds have trivial tangent bundle, being smooth affine varieties of odd dimension.

Part A is messier, but possible: The faces of P and the subtori of T^n form a cell decomposition from which you can calculate everything in terms of the polytope. Actually there is a nice splitting of the homology groups in terms of relative homology groups of pairs formed by the polytope and unions of its facets.

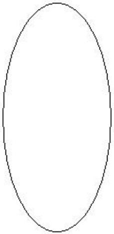
But it is well-known that this is not enough to determine the manifold, the earliest example being that of the exotic spheres: the homotopy and tangential data are the same for all homotopy spheres.

Starting from this case it was established that to understand a manifold M it is necessary to find a manifold Q whose boundary is M and study the questions A and B above for Q. If all goes well, one can determine the smooth type of Q (in many cases with the help of the h-cobordism theorem) a therefore also that of M. For a homotopy sphere one tries first to see what kind of manifold it bounds (there is always one). If it does not bound a parallelizable manifold then it is not standard. If it does, one studies the invariants of that manifold to see which sphere is its boundary.

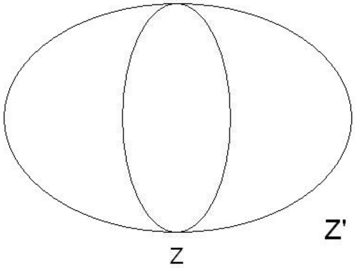
In our case is easy to find such manifold for Z: Add one more real variable x_0 and duplicate the first coefficient to obtain a close manifold Z' of one more dimension:

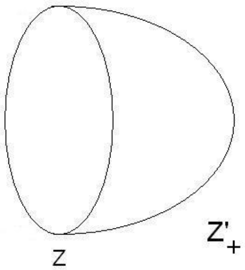
$$\lambda_1 x_0^2 + \lambda_1 |z_1|^2 + \Sigma_{i=2}^n \lambda_i |z_i|^2 = 0$$
$$x_0^2 + |z_1|^2 + \Sigma_{i=2}^n |z_i|^2 = 1$$

Z is the intersection of this manifold with the hyperplane $x_0 = 0$ and if we intersect Z' with the half space $x_0 \geq 0$ we obtain a manifold Z'_+ whose boundary is Z.



Z


Z'
Z


Z
Z'+

Now we have to repeat the steps A and B above for Z'_+: it is different from Z in two ways:

It is only *half* a manifold and even the whole manifold is different:

$$\lambda_1 x_0^2 + \Sigma\lambda_i |z_i|^2 = 0$$
$$x_0^2 + \Sigma |z_i|^2 = 1$$

It has one *real variable* so it does not fit in the above scheme.

This practically obliged me to start all over again with the more general analogous situation with real coordinates:

So now Z will be now the manifold in $\mathbb{R}^n$ given by the equations

$$\Sigma\lambda_i x_i^2 = 0$$
$$\Sigma x_i^2 = 1$$

Which is the real part of the previous manifold, which we will now denote by $Z^{\mathbb{C}}$:

$$\Sigma\lambda_i \left|z_i\right|^2 = 0$$
$$\Sigma \left|z_i\right|^2 = 1$$

[I wonder why I call them now Z's,
......................they used to be M's (for manifolds).]

Z is again smooth under the same hypothesis, it's tangent bundle is not necessarily trivial but only *stably* trivial, which is good enough. As before, it bounds a parallelizable manifold Z'_+.

Now it has a $\mathbb{Z}_2^n$ action with the same quotient P as before. And Z can be recovered from P just as above, but in an even more geometrical way:

P lies in $(\mathbb{R}_+)^n$ and one obtains Z by reflecting P successively in all the hyperplanes $x_i = 0$.

In the triangular case $n = p + q + r$ we get again a triple product of spheres, each of which can now have even or odd dimension: $S^{p-1} \times S^{q-1} \times S^{r-1}$

For the pentagon, Z is a surface and one can compute its Euler characteristic:

Z is formed by:

32 pentagons, $5 \times 16 = 80$ edges and $5 \times 8 = 40$ vertices.

So the Euler characteristic is -8 ...

...and we conclude that it is the surface of genus 5:

$$Z = (S^1 \times S^1)\#(S^1 \times S^1)\#(S^1 \times S^1)\#(S^1 \times S^1)\#(S^1 \times S^1)$$

The fact that the old manifold $Z^{\mathbb{C}}$ corresponding to the pentagon is

$$(S^3 \times S^4)\#(S^3 \times S^4)\#(S^3 \times S^4)\#(S^3 \times S^4)\#(S^3 \times S^4)$$

is no longer surprising!!

With the same natural idea one can compute the homology groups:

The faces of P and all their reflections form a cell decomposition of Z from which I could compute the homology groups. In fact the result and the proof are very geometric:

$$H_i(Z) \cong \bigoplus_J H_i(P, P_J)$$

[no shift in dimensions!] where the sum is over all subsets $J \subset \{1, \ldots, n\}$ and P_J is the union of all facets of P corresponding to $i \in J$.

This splitting is very explicit geometrically, is valid at the chain level and it uses explicitly the action of $\mathbb{Z}_2^n$, *so it includes the information of the action of this group on the homology.*

- Then I had to do all over again the geometry and the splitting for the homology of the manifold with boundary Z'_+.

  With that, and a few more technical lemmas, I could prove that the manifold with boundary Z' was exactly the one that one could expect and so is then the manifold Z:

- It is either a triple product of spheres (in the triangular case), or

- It is a connected sum of sphere products $S^{a_i} \times S^{b_i}$ (in all the other cases), where the number of summands and a_i, b_i are determined by the combinatorial data.

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There is a problem with the proof: the use of the h-cobordism theorem works only if the manifold Z'_+ is simply connected and of dimension at least 6.

This implies that the above result was proved only under those assumptions, which exclude some of the pentagonal cases and one heptagonal one, for all of which the result is surely also true.
But for the original question, the topology of the manifolds $Z^{\mathbb{C}}$, this causes no problem: the proof covers them all and so they are all triple products or connected sums.

The next step would have been the topological description of the intersection of more quadrics:

$$\Sigma \Lambda_i x_i^2 = 0$$

$$\Sigma x_i^2 = 1$$

where $\Lambda_i \in \mathbb{R}^k$.

- Most of the parts of the above proof work:
- The regularity of the system is equivalent to a higher dimensional version of weak hyperbolicity.
- Every Z is equivalent to one obtained from a *primitive* configuration with multiplicities.
- Z is always stably parallelizable and bounds a parallelizable manifold.
- There is the $\mathbb{Z}_2^n$ action with a simple polytope as quotient-section. The polytope can be any simple polytope P and determines completely the manifold (and the action).
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- The only obstacle to obtaining a general theorem was the difficulty of describing all the *primitive* configurations. If these were the odd regular polytopes for $k = 2$, now they would be certain constellations in the unit sphere S^{k-1} whose enumeration, even in the case $k = 3$, seemed difficult.

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- All this I knew essentially by the end of 1984, although some of the details of the proof were checked later.

- In 1986 I learned that there was some previous work on the subject: Marc Chaperon had studied the same objects, as "pseudo-quotients" of actions of $\mathbb{R}^n$ or $\mathbb{C}^n$, Hirzebruch had described all the 2-dimensional ones which appeared to him in a problem of Algebraic Geometry; and C.T.C. Wall had studied the case $k = 2$ as above from questions in the Theory of Singularities of Maps). From him I learned that the correspondence between configurations Λ_i and convex polytopes was known as the *Gale Transform*.

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- But it affected my projects, especially in combinatorics: my description of all the possibilities for $k = 2$ was not far from a classification of all d dimensional polytopes with at most $d + 3$ facets. So in 1986 I learned that this had already been done and that the analog result for $d + 4$ facets was extremely difficult.

- So I published in 1987 my main results for $k = 2$ with just all too brief remarks about which parts of the proof were valid for any k. There were also the techniques used and the new group actions (including not only those of the torus and $\mathbb{Z}_2^n$, but also the action of the symmetry group of P), but there was no reaction from the topologists.

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- These turn out to be interesting examples of complex, non-symplectic manifolds, that come equipped with natural deformation spaces and foliations.

- Further quotients produce many toric varieties, including all smooth ones.

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- In 2000 Bosio and Meersseman made a very deep and thorough study of those complex manifolds, that also included remarkable advances on the topology of the manifolds $Z^{\mathbb{C}}$ themselves, among them:

- a) They showed that for $k \geq 3$ the manifold $Z^{\mathbb{C}}$ can be considerably more complicated than what was known for the case $k = 2$. But they also proved than that under certain conditions one obtains again the same kind of connected sums. They also conjectured that the same would be true for a wide family.
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INTERSECTION OF QUADRICS.

C.T.C. Wall.

Hirzebruch.

Camacho-Kuiper-Palis

Chaperon

LdM

Loeb-Nicolau

Verjovsky

Meersseman

Bosio

Seade...Pichon

Cisneros, Oka

Vinicio Gómez

Mathilde Kammerer

Chaperon-LdM

Seade, Gómez Mont, Ruas, Verjovsky, Pichon, Cisneros, Oka, Vinicio Gómez have worked on some manifolds given by polynomial equations on z_i, $\bar{z}_i$, in general of higher degree, except for Vinicio who has studied intersections of quadrics with crossed terms ($z_i \bar{z}_j$ with $i \neq j$).

Mathilde Kammerer, Chaperon and myself proved that all $Z^{\mathbb{C}}$ manifolds can be born stably as invariant manifolds in the bifurcation of dynamical systems, in the same way as S^1 is born in the Hopf bifurcation.

But this is not the end of the story.

In October 2007 I heard a talk by Sam Gitler about the *Generalized Moment-Angle Complexes* and the *Polyhedral product functor.*

I did not understand anything, but everything was very familiar:

There was a polyhedron K that dictated how to construct certain spaces, whose homology was then computed through a splitting determined by certain subpolyhedra K_J, there were group actions and ...

Through this I discovered that there was a whole school that had been studying for decades (among many other objects) my dear old manifolds under the very strange names of *Moment-angle manifolds* (the complex ones) and *Universal abelian covers* (the real ones).

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MOMENT-ANGLE COMPLEXES.

Demazure
...
Atiyah
Guillemin-Sternberg
Delzant, Audin
...
Davis-Januzskiewicz
...
Buchstaber-Panov
Franz, Masuda, Ray
...
Denham-Suciu
Bahri-Bendersky-
-Cohen-Gitler.

The dots on the right hand column reflect mainly my ignorance of what is now called Toric Topology. Of course, the list itself reflects my own interpretation of a story I know not well enough, hoping that those directly involved will correct me...

Of course we knew about toric manifolds and Delzant's actions (and Meersseman-Verjovsky established a relation between quadrics and those objects explicitly). Yet, as a real connection between the two schools, Bosio-Meersseman and Denham-Suciu are, to my knowledge, the first on each side of the list to quote the work of someone on the other. In the case of Bosio-Meersseman, they quote mainly the previous computation of the cohomology ring of $Z^{\mathbb{C}}$ by Buchstaber-Panov.

A curious fact is that, although the same objects appear on both sides, most of the main results are very different and actually complementary. Much work is devoted in both sides to different quotients of the common objects: the complex LV-M-B manifolds on one side and the (quasi-)toric manifolds on the other.

In the quadrics side we were overly optimistic and thought for sometime that all the $Z^{\mathbb{C}}$ manifolds were nice and simple, until Bosio-Meersseman showed us that it was not so, and yet we keep looking for the nice and simple ones up to this day. In the moment-angle side they knew some really bad examples, far more complicated than the any of Bosio-Meersseman or than our worst nightmares, but somehow missed most of the nice, simple ones.

Some things are a lot simpler from the quadrics point of view: for example, the one-line proof above that the manifolds Z (and therefore all the $Z^{\mathbb{C}}$ which are a subfamily) bound parallelizable manifolds. Compare with the proof in Buchstaber-Panov that they bound. I am sure that they would have proved without any problem that they bound parallelizable manifolds if they had found it necessary, for me it was an essential ingredient of the proofs.

Of the things proved from the moment-angle complexes side, some simple ones I can see immediately in the context of quadrics. But most of them I still have not the faintest idea of how to prove using the quadrics approach.

For example, the splitting of the suspension of the moment-angle complexes shows that, in the case of manifolds, the homology fundamental class becomes spherical after one suspension, a fact that I do not know how to prove otherwise.

This fact suggests (but does not prove) that all the moment-angle manifolds could be hypersurfaces, and in fact all the ones that we know explicitly are so. One could try to prove this geometrically, which would give a new proof of sphericity of the suspension of the fundamental class.

Of course the quadrics approach does not produce results about moment-angle complexes which are not manifolds. Nevertheless, the results (and crucial examples) about the special case give ideas of what to expect (or not) in the general setting.

- Continuing with my story, Sam and I got into an intense process of discussion through which I began to understand slowly the relations between their abstract functorial construction and my very concrete varieties with their second degree equations.

- Finally I managed to see that all my old manifolds (real and complex, closed or with boundary... or even with corners, if needed) and the constructions on them (Z_0, Z', Z_+, multiplicities, etc.) fitted inside this most abstract framework, but actually not in any of the previous ones.

- And there was this fantastic geometric splitting for moment-angle complexes that allowed, among many other things, the computation of the homology groups of all the old objects once and for all. This allowed us to get a global view of all the objects and their interrelations which, once spotted, could be naturally proved.

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- With this vision we looked at the Bosio-Meersseman conjecture, and we managed to prove much more.

- We proved that if P is an even dimensional *neighborly* polytope then Z is a connected sum of sphere products and so is any manifold Z^J obtained from Z by introducing multiplicities.

- And that means *a lot* of examples: there are many *neighborly* polytopes in each dimension (it is claimed that in a certain sense *most* polytopes are *neighborly*) and for each of them there is an infinite family P^J all of which are not, and they all produce manifolds that are connected sums of sphere products.

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- Then we looked at another question raised by Bosio and Meersseman: the transition between types of manifolds when the weak hyperbolicity condition is broken.

- We analyzed what happens when the polytope P is altered by cutting off a vertex or an edge. We obtained a good result for the manifolds Z, thus giving in particular new infinite families of connected sums of sphere products.

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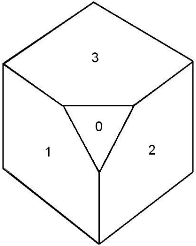
The result for the manifolds $Z^{\mathbb{C}}$ is limited, but good enough to understand another question of Bossio and Meersseman concerning a very interesting example of them: the truncated cube.



- By rather elementary methods we showed that the corresponding manifolds are:

$$Z = 2(S^1 \times S^1 \times S^1)\#7(S^2 \times S^1)$$

$$Z^{\mathbb{C}} = \mathcal{G}(S^3 \times S^3 \times S^3)\#3(S^3 \times S^7)\#3(S^4 \times S^6)\#(S^5 \times S^5)$$

(here $\mathcal{G}(M)$ is a simple operation: remove a disk from M, multiply times D^2 and take the boundary.)

- These two manifolds have the same homology groups, if one forgets the grading. This is consistent with the splitting results. It was expected that the cohomology rings would also be, at least with $\mathbb{Z}_2$ coefficients, since there are questions of sign in different dimensions.

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- For a long time we tried to prove that it was so, without success. Finally we turned our failure into a proof that they are in fact *not* isomorphic as ungraded rings with $\mathbb{Z}_2$ coefficients.

- This is very surprising, contradicts some published results and shows that a very different rule applies for the cup product of the Z and the $Z^{\mathbb{C}}$ manifolds. The proof is actually quite simple and elementary and can be easily checked by anyone: just verify the elementary constructions that give the topological type of Z and verify the algebraic proof that the two rings are not isomorphic. (The more involved construction giving the topological type of $Z^{\mathbb{C}}$ need not be checked, its cohomology ring was computed by Bosio and Meersseman and also follows directly from the well-known results on the cohomology ring of moment-angle manifolds).

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- Conclusions:

- 1.- I deeply appreciate this opportunity to talk in front of my old and my new friends about my old work, which is not well known. As you have seen, there were some things which I discovered, rediscovered or *prediscovered*.

- 2.- But my real point is not to say *"I did this first"* or *"I can do this better"* or *"I can do this faster"*. (And it is quite messy to elucidate in this complicated story who said what first, in which context and with what generality).

- My real point is that I have done some things *differently* and that it is very useful to have two different points of view and to combine them in a given problem. This happened in my recent work with Sam which combined the concrete topological methods and the abstract functorial approach.

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