Asymptotics of the interface of Laplacian growth with multiple point sources

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Asymptotic shape of the free boundary

Hele-Shaw flows

Hele-Shaw flows Weak solutions and Quadrature domains Main result and its proof

 $\Omega(0)$: initial domain $c_1, \ldots, c_l \in \Omega(0)$: injection points $\alpha_1, \ldots, \alpha_l > 0$: injection rates



Hele-Shaw flows Weak solutions and Quadrature domains Main result and its proof

Hele-Shaw flows

$$\left\{ egin{array}{ccc} (1) & -\Delta p = \sum_{j=1}^l lpha_j \delta_{c_j} & ext{in } \Omega(t) \ (2) & p = 0 & ext{on } \partial \Omega(t) \ (3) & -\partial_n p = v_n & ext{on } \partial \Omega(t) \end{array}
ight.$$

p(z,t): pressure of fluid $\Omega(t)$: fluid domain at time $t \ge 0$ δ_{c_j} : the Dirac measure

 $V = -\nabla p$: velocity field



Hele-Shaw flows Weak solutions and Quadrature domains Main result and its proof

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$$(1),(2) \Rightarrow p(z,t) = \sum_{j=1}^{l} \alpha_j G_{c_j,\Omega(t)}(z)$$

 $(G_{c_j,\Omega(t)}:$ Green's function for $-\Delta)$

Hele-Shaw flows Weak solutions and Quadrature domains Main result and its proof

Hele-Shaw flows

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$$(1),(2) \Rightarrow p(z,t) = \sum_{j=1}^{l} \alpha_j G_{c_j,\Omega(t)}(z)$$

Find $\{\Omega(t)\}_{t>0}$ s.t.

$$-\sum_{j=1}^l lpha_j rac{\partial G_{c_j,\Omega(t)}}{\partial n} = v_n \quad ext{on } \partial \Omega(t).$$

Weak solutions

Assume
$$-\sum_{j=1}^{l} \alpha_j \frac{\partial G_{c_j,\Omega(t)}}{\partial n} = v_n \quad \text{on } \partial \Omega(t).$$

$$\int_{\Omega(t)\setminus\Omega(0)} s(z)\,dm$$

Weak solutions

Assume
$$-\sum_{j=1}^{l} lpha_j rac{\partial G_{c_j,\Omega(t)}}{\partial n} = v_n \quad ext{on } \partial \Omega(t).$$

$$\int_{\Omega(t)\setminus\Omega(0)} s(z)\,dm = \int_0^t \int_{\partial\Omega(au)} s(z)\cdot v_n\,d\sigma\,d au$$

Weak solutions

Assume
$$-\sum_{j=1}^{l} \alpha_j \frac{\partial G_{c_j,\Omega(t)}}{\partial n} = v_n \text{ on } \partial \Omega(t).$$

$$\begin{split} \int_{\Omega(t)\setminus\Omega(0)} s(z) \, dm &= \int_0^t \int_{\partial\Omega(\tau)} s(z) \cdot v_n \, d\sigma \, d\tau \\ &= \sum_{j=1}^l \alpha_j \int_0^t \int_{\partial\Omega(\tau)} s(z) \cdot \left(-\frac{\partial G_{c_j,\Omega(\tau)}}{\partial n} \right) \, d\sigma \, d\tau \end{split}$$

Weak solutions

Assume
$$-\sum_{j=1}^{l} \alpha_j \frac{\partial G_{c_j,\Omega(t)}}{\partial n} = v_n$$
 on $\partial \Omega(t)$.

$$\begin{split} \int_{\Omega(t)\setminus\Omega(0)} s(z) \, dm &= \int_0^t \int_{\partial\Omega(\tau)} s(z) \cdot v_n \, d\sigma \, d\tau \\ &= \sum_{j=1}^l \alpha_j \int_0^t \int_{\partial\Omega(\tau)} s(z) \cdot \left(-\frac{\partial G_{c_j,\Omega(\tau)}}{\partial n} \right) \, d\sigma \, d\tau \\ &\geq \sum_{j=1}^l \alpha_j \int_0^t s(c_j) \, d\tau \end{split}$$

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ight)\,d\sigma\,d\tau \ &\geq \sum_{j=1}^l lpha_j \int_0^t s(c_j)\,d au \ = t \sum_{j=1}^l lpha_j s(c_j). \end{aligned}$$

Hele-Shaw flows Weak solutions and Quadrature domains Main result and its proof

Weak solutions

$$(st) \left\{ egin{array}{l} \int_{\Omega(0)} s \, dm + t \sum_{j=1}^l lpha_j s(c_j) \leq \int_{\Omega(t)} s \, dm \ (orall s \in SL^1(\Omega(t))), \ m(\Omega(t)) < \infty. \end{array}
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Weak solutions

Weak solution

For each t > 0, find $\Omega(t)$ s.t.

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$$(*) \begin{cases} \int_{\Omega(0)} s \, dm + t \sum_{j=1}^{l} \alpha_j s(c_j) \leq \int_{\Omega(t)} s \, dm \\ (\forall s \in SL^1(\Omega(t))), \\ m(\Omega(t)) < \infty. \end{cases}$$

existence of weak solution [Sakai (1982)] Let $\Omega(0)$ be a domain with $m(\Omega(0)) < \infty$. Then, there exists a domain $\Omega(t)$ satisfying (*).

 $\begin{array}{c|c} \mbox{uniqueness of weak solution [Sakai (1982)]} \\ \mbox{If } \Omega(t) \mbox{ and } \Omega(t)' \mbox{ satisfy (*), then } \chi_{\Omega(t)} = \chi_{\Omega(t)'} \quad m\mbox{-}a.e. \end{array}$

Weak solutions

Weak solution

Quadrature Domain of
$$t\sum lpha_j \delta_{c_j}$$

For each t > 0, find $\Omega(t)$ s.t.

$$(*) \begin{cases} \int_{\Omega(0)} s \, dm + t \sum_{j=1}^{l} \alpha_j s(c_j) \leq \int_{\Omega(t)} s \, dm \\ (\forall s \in SL^1(\Omega(t))), \\ m(\Omega(t)) < \infty. \end{cases}$$

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Hele-Shaw flows Weak solutions and Quadrature domains Main result and its proof

Known result [Sakai (1998)]

Let $\Omega(0) \subset D(c,r)$ and $m(\Omega(0)) + t \sum_{j=1}^{l} \alpha_j \geq 4\pi r^2$. Then,

$$D({\color{black}c}, ilde{R}(t)-{\color{black}r})\subset \Omega(t)\subset D({\color{black}c}, ilde{R}(t)+{\color{black}r})$$

holds, where
$$ilde{R}(t):=\sqrt{rac{1}{\pi}\left(m(\Omega(0))+t\sum_{j=1}^{l}lpha_{j}
ight)}.$$



Theorem

Let
$$\Omega(0) \subset D(c, r)$$
. Then, for sufficiently large $t > 0$,
 $D\left(\frac{w_l}{l}, R(t) - \varepsilon_l^{-}(t)\right) \subset \Omega(t) \subset D\left(\frac{w_l}{l}, R(t) + \varepsilon_l^{+}(t)\right)$

holds, where

$$w_l := rac{\sum_{j=1}^l lpha_j c_j}{\sum_{j=1}^l lpha_j}, \; R(t) := \sqrt{rac{t}{\pi} \sum_{j=1}^l lpha_j},$$

Theorem

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$$\begin{split} \varepsilon_{l}^{-}(t) &:= \sqrt{\frac{\pi}{\sum_{k=1}^{l} \alpha_{k}}} \left(\sum_{j=2}^{l} \frac{\alpha_{j} \sum_{k=1}^{j-1} \alpha_{k}}{\left(\sum_{k=1}^{j} \alpha_{k} \right)^{2}} \left| w_{j-1} - c_{j} \right|^{2} \right) t^{-1/2} + O(t^{-1}), \\ \varepsilon_{l}^{+}(t) &:= \sqrt{\frac{\pi}{\sum_{k=1}^{l} \alpha_{k}}} \left(\sum_{j=2}^{l} \frac{\alpha_{j} \sum_{k=1}^{j-1} \alpha_{k}}{\left(\sum_{k=1}^{j} \alpha_{k} \right)^{2}} \left| w_{j-1} - c_{j} \right|^{2} + \frac{r^{2}}{2} \right) t^{-1/2} \\ &+ O(t^{-1}). \end{split}$$

Theorem

Let
$$\Omega(0) \subset D(c, r)$$
. Then, for sufficiently large $t > 0$,
 $D\left(\frac{w_l}{k}, R(t) - \varepsilon_l^{-}(t)\right) \subset \Omega(t) \subset D\left(\frac{w_l}{k}, R(t) + \varepsilon_l^{+}(t)\right)$

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Outline of Proof

Recall the Hele-Shaw problem: Find $\Omega(t)$ s.t.

$$\int_{\Omega(0)} s\,dm + t\sum_{j=1}^l lpha_j s(c_j) \leq \int_{\Omega(t)} s\,dm \quad ig(orall s\in SL^1(\Omega(t))ig)\,.$$

Outline of Proof

Recall the Hele-Shaw problem: Find $\Omega(t)$ s.t.

$$\int_{\Omega(0)} s \, dm + t \sum_{j=1}^{l} \alpha_j s(c_j) \leq \int_{\Omega(t)} s \, dm \quad (\forall s \in SL^1(\Omega(t)))$$

Outline of Proof

the Schwarz function defined the Schwarz function of $\partial \Omega_0(t)$.

If we have a holomorphic func. in a nb'd of $\partial\Omega_0(t)$ satisfying

(i) $S(z) = \overline{z}$ on $\partial \Omega_0(t)$;

the Schwarz function the Schwarz function of $\partial \Omega_0(t)$.

If we have a holomorphic func. in a nb'd of $\partial\Omega_0(t)$ satisfying

- (i) $S(z) = \overline{z}$ on $\partial \Omega_0(t)$;
- (ii) S is meromorphic in $\Omega_0(t)$ and has simple poles at
 - c_1,\ldots,c_l with respective residues $t\alpha_1/\pi,\ldots,t\alpha_l/\pi$,

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then, $\forall f$: holomorphic in a nb'd of $\overline{\Omega_0(t)}$,

$$egin{aligned} &\int_{\Omega_0(t)} f\,dm = rac{1}{2i}\int_{\partial\Omega_0(t)} f(z)\overline{z}\,dz \ &= rac{1}{2i}\int_{\partial\Omega_0(t)} f(z)S(z)\,dz = t\sum_{j=1}^llpha_jf(c_j). \end{aligned}$$

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If we have a holomorphic func. in a nb'd of $\partial\Omega_0(t)$ satisfying

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Hence, $\Omega_0(t)$ is expected to be the desired q. d.

Outline of Proof

• Case
$$l = 2$$

(a) construction of $\Omega_0(t)$ and its estimate
(b) holomorphic class $\rightarrow SL^1$

$$\Omega\left(\sum_{j=1}^{l}\nu_{j}\right) = \Omega\left(\chi_{\Omega\left(\sum_{j=1}^{l-1}\nu_{j}\right)} + \nu_{l}\right)$$

Outline of Proof

1 Case
$$l = 2 \implies c_1 = i, c_2 = -i, \alpha_1 = 1$$

(a) construction of $\Omega_0(t)$ and its estimate
(b) holomorphic class $\rightarrow SL^1$

$$\Omega\left(\sum_{j=1}^{l}\nu_{j}\right) = \Omega\left(\chi_{\Omega\left(\sum_{j=1}^{l-1}\nu_{j}\right)} + \nu_{l}\right)$$

Example. (the Schwarz function of $\partial D(0,1)$)

$$S(z):=rac{1}{z}.$$

Note that $R(z) := \overline{S(z)} = z/|z|^2$ is the reflection associated to the unit circle $\partial D(0, 1)$.

$$\bigcap_{D(0,1)}^{R}$$

Hele-Shaw flows Weak solutions and Quadrature domains Main result and its proof

(a) construction of $\Omega_0(t)$ and its estimate

We define

 $\Omega_0(t) := \varphi\left(D(0,1)\right),$

where

$$arphi(z):=rac{ab(z-ic)}{z^2+b^2}+ibc, \hspace{1em} R(z)=1/\overline{z}.
onumber \ (a(t),\ b(t),\ c(t): ext{ parameters})$$



We define

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 $\Rightarrow \bullet S \text{ is the Schwarz function of } \partial \Omega_0(t) \text{, i.e., } S \text{ satisfies (i).}$ • Choose a, b, c, appropriately so that S satisfies (ii).

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⇒ • S is the Schwarz function of $\partial \Omega_0(t)$, i.e., S satisfies (i). • Choose a, b, c, appropriately so that S satisfies (ii). • $a(t) = a_1t + a_2 + \cdots$, $b(t) = b_1t^{1/2} + b_2t^{-1/2} + \cdots$, $c(t) = c_1t^{1/2} + c_2t^{-1/2} + \cdots$.

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⇒ • S is the Schwarz function of $\partial \Omega_0(t)$, i.e., S satisfies (i). • Choose a, b, c, appropriately so that S satisfies (ii). • $a(t) = a_1t + a_2 + \cdots$, $b(t) = b_1t^{1/2} + b_2t^{-1/2} + \cdots$, $c(t) = c_1t^{1/2} + c_2t^{-1/2} + \cdots$. • Estimate $|\varphi(z) - w_l|$, $z \in \partial D(0, 1)$.

Outline of Proof

• Case
$$l = 2$$

(a) construction of $\Omega_0(t)$ and its estimate
(b) holomorphic class $\rightarrow SL^1$

$$\Omega\left(\sum_{j=1}^{l}\nu_{j}\right) = \Omega\left(\chi_{\Omega\left(\sum_{j=1}^{l-1}\nu_{j}\right)} + \nu_{l}\right)$$

Hele-Shaw flows Weak solutions and Quadrature domains Main result and its proof

$$egin{aligned} &\Omega\left(t\sum_{j=1}^{3}lpha_{j}\delta_{c_{j}}
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Hele-Shaw flows Weak solutions and Quadrature domains Main result and its proof

$$\begin{split} \Omega\left(t\sum_{j=1}^{3}\alpha_{j}\delta_{c_{j}}\right) \\ &= \Omega\left(\chi_{\Omega\left(t(\alpha_{1}\delta_{c_{1}}+\alpha_{2}\delta_{c_{2}})\right)} + t\alpha_{3}\delta_{c_{3}}\right) \\ &\approx \Omega\left(\chi_{D\left(\frac{\alpha_{1}c_{1}+\alpha_{2}c_{2}}{\alpha_{1}+\alpha_{2}},\sqrt{\frac{t}{\pi}(\alpha_{1}+\alpha_{2})}\right)} + t\alpha_{3}\delta_{c_{3}}\right) \\ &= \Omega\left(t(\alpha_{1}+\alpha_{2})\delta_{\frac{\alpha_{1}c_{1}+\alpha_{2}c_{2}}{\alpha_{1}+\alpha_{2}}} + t\alpha_{3}\delta_{c_{3}}\right) \\ &\approx D\left(\frac{\alpha_{1}c_{1}+\alpha_{2}c_{2}+\alpha_{3}c_{3}}{\alpha_{1}+\alpha_{2}+\alpha_{3}},\sqrt{\frac{t}{\pi}\left(\sum_{j=1}^{3}\alpha_{j}\right)}\right) \end{split}$$

Stability of the free boundary

Derivation of an evolution equation

$$\begin{cases} (1) & -\Delta p = \alpha_1 \delta_i + \alpha_2 \delta_{-i} & \text{in } \Omega(t) \\ (2) & p = 0 & \text{on } \partial \Omega(t) \\ (3) & -\partial_n p = v_n & \text{on } \partial \Omega(t) \end{cases}$$



Q. What if the initial domain $\Omega(t_0)$ is close enough to $\Omega_0(t_0)$?

Derivation of an evolution equation Main result and its proof

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Framework

For
$$r \in C(\partial D)$$
, set
 $\partial D_r := \{(1 + r(\xi))\xi \mid \xi \in \partial D\}.$



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For
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- $(p, \{\Omega(t)\})$ \downarrow Assume $\Omega(t) = \varphi_t(D_{r(\cdot,t)}).$
- r : a non-local evolution equation



Derivation of an evolution equation Main result and its proof

Derivation of an evolution equation

$$egin{aligned} \partial_t r &= rac{-\langle
abla p \circ arphi, (D_w arphi) n
angle - \langle \partial_t arphi, (D_w arphi) n
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Theorem

Evolution equation in $h^{1,\gamma}(\partial D)$

$$\left\{ egin{array}{ll} r'=\mathcal{F}(r,t), & t>t_0, \ r(t_0)=r_0\in h^{2,\gamma}(\partial D), \end{array}
ight.$$

where $\mathcal{F}(\cdot,t): h^{2,\gamma}(\partial D) \to h^{1,\gamma}(\partial D), \ \mathcal{F}(0,t) = 0.$

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where $\mathcal{F}(\cdot,t): h^{2,\gamma}(\partial D) \to h^{1,\gamma}(\partial D), \ \mathcal{F}(0,t) = 0.$

The little Hölder space

$$h^{k,\gamma}(\partial D):=\overline{C^\infty(\partial D)}^{C^{k,\gamma}(\partial D)}.$$

Theorem

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where $\mathcal{F}(\cdot,t): h^{2,\gamma}(\partial D) \to h^{1,\gamma}(\partial D), \ \mathcal{F}(0,t) = 0.$

Theorem

Suppose t_0 is sufficiently large. For $\varepsilon > 0$, there are $\delta, M > 0$ s.t. if $||r_0||_{h^{2,\gamma}} < \delta$, then there exists a unique solution $r \in C([t_0,\infty); h^{2,\gamma}) \cap C^1([t_0,\infty); h^{1,\gamma})$ satisfying

$$||r(t)||_{h^{2,\gamma}} + t ||r'(t)||_{h^{1,\gamma}} \le M t^{-1+\varepsilon} ||r_0||_{h^{2,\gamma}}.$$

Evolution equation in $h^{1,\gamma}$

$$\left\{ \begin{array}{l} r'=\mathcal{F}(r,t), \quad t>t_0,\\ r(t_0)=r_0\in h^{2,\gamma}, \end{array} \right.$$

$$\begin{split} \mathcal{F}(\cdot,t) &: h^{2,\gamma} \to h^{1,\gamma}, \\ \mathcal{F}(0,t) &= 0. \end{split}$$

Lemma 1 $| t \cdot D_r \mathcal{F}(r,t) \to \mathcal{A}$ as $t \to \infty, r \to 0$.

Evolution equation in $h^{1,\gamma}$

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Lemma 1 $t \cdot D_r \mathcal{F}(r,t)
ightarrow \mathcal{A}$ as $t
ightarrow \infty, \ r
ightarrow 0.$

 $\begin{array}{c|c} \hline \text{Lemma 2} & \mathcal{A}: \text{ sectorial in } h^{1,\gamma}, \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(\mathcal{A})\} = -1.\\ & (\text{'97 Escher \& Simonett, '09 Vondenhoff}) \end{array}$

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Lemma 2 \mathcal{A} : sectorial in $h^{1,\gamma}$, $\sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(\mathcal{A})\} = -1$. Set $\tau := \log(t/t_0)$ and $\tilde{r}(\tau) := r(t)$, then (P) $\tilde{r}'(\tau) = t \cdot r'(t) = t \cdot \mathcal{F}(r,t) = \mathcal{A}\tilde{r} + \mathcal{G}(\tilde{r},\tau)$, where $\mathcal{G}(0,\tau) = 0$, $D_{\tilde{r}}\mathcal{G}(\tilde{r},\tau) \to 0$ as $t \to \infty$, $\tilde{r} \to 0$.

Evolution equation in $h^{1,\gamma}$

$$\left\{ egin{array}{ll} r'=\mathcal{F}(r,t), & t>t_0, & \mathcal{F}(\cdot,t), \\ r(t_0)=r_0\in h^{2,\gamma}, & \mathcal{F}(0,t) \end{array}
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= 0.

Outline of Proof

Evolution equation in $h^{1,\gamma}$

$$\left(egin{array}{ll} r'=\mathcal{F}(r,t), & t>t_0, & \mathcal{F}(\cdot,t):h^{2,\gamma}
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ight.$$

Lemma $1 \mid t \cdot D_r \mathcal{F}(r,t) \to \mathcal{A}$ as $t \to \infty, r \to 0$.

Lemma 2 $| \mathcal{A} :$ sectorial in $h^{1,\gamma}$, sup{Re $\lambda | \lambda \in \sigma(\mathcal{A})$ } = -1. Set $\tau := \log(t/t_0)$ and $\tilde{r}(\tau) := r(t)$, then (P) $\tilde{r}'(\tau) = t \cdot r'(t) = t \cdot \mathcal{F}(r,t) = \mathcal{A}\tilde{r} + \mathcal{G}(\tilde{r},\tau),$ where $\mathcal{G}(0,\tau) = 0$, $D_{\tilde{r}}\mathcal{G}(\tilde{r},\tau) \to 0$ as $t \to \infty$, $\tilde{r} \to 0$. $\mathcal{A}: h^{2,\gamma} o h^{1,\gamma}, \, \mathcal{G}(\cdot, au): h^{2,\gamma} o h^{1,\gamma}.$ $\tilde{r} \in C([0,\infty); h^{2,\gamma})$ $\Rightarrow \tilde{r} \in C([0,\infty); h^{2,\gamma})?$

Evolution equation in $h^{1,\gamma}$

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Evolution equation in $h^{1,\gamma}$

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 $\begin{array}{c} \mbox{Lemma 2} \end{array} \mathcal{A} : \mbox{sectorial in } h^{1,\gamma}, \mbox{sup} \{ \mbox{Re}\lambda \mid \lambda \in \sigma(\mathcal{A}) \} = -1. \\ \mbox{Set } \tau := \log(t/t_0) \mbox{ and } \tilde{r}(\tau) := r(t), \mbox{ then} \\ (\mathbb{P}) \quad \tilde{r}'(\tau) = t \cdot r'(t) = t \cdot \mathcal{F}(r,t) = \mathcal{A}\tilde{r} + \mathcal{G}(\tilde{r},\tau), \\ \mbox{where } \mathcal{G}(0,\tau) = 0, \ D_{\tilde{r}}\mathcal{G}(\tilde{r},\tau) \to 0 \mbox{ as } t \to \infty, \ \tilde{r} \to 0. \\ \mbox{\mathcal{A}} : h^{2,\gamma} \to h^{1,\gamma}, \ \mathcal{G}(\cdot,\tau) : h^{2,\gamma} \to h^{1,\gamma}. \\ \tilde{r} \in C([0,\infty); h^{2,\gamma}) \Rightarrow \mathcal{G}(\tilde{r},\tau) \in C([0,\infty); h^{1,\gamma}) \\ \Rightarrow \tilde{r} \in C([0,\infty); h^{2,\gamma}) \\ (\because \mbox{ maximal regularity in } h^{k,\gamma} \ ('79 \mbox{ Da Prato \& Grisvard})). \end{array}$

Evolution equation in $h^{1,\gamma}$

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Evolution equation in $h^{1,\gamma}$

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Evolution equation in $h^{1,\gamma}$

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 $\therefore \|r(t)\|_{h^{2,\gamma}} + t \|r'(t)\|_{h^{1,\gamma}} \le M t^{-1+\varepsilon} \|r_0\|_{h^{2,\gamma}}.$