Non-univalent solutions of the Polubarinova - Galin equation

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Joint work with

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PG: $\mathrm{Re}\left[\dot{f}(\xi,t)\,\overline{\xi f'(\xi,t)}\right] = q(t)$

$(\xi \in \partial\mathbb{D})$

LK: $\dot{f}(\xi,t) = \xi f'(\xi,t)\, P(\xi,t) \quad (\xi \in \mathbb{D})$

$$P(\xi,t) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{q(t)}{|f'(z,t)|^2} \cdot \frac{z+\xi}{z-\xi}\,\frac{dz}{z}$$

Normalization $\quad f(0,t) = 0, \quad f'(0,t) > 0.$

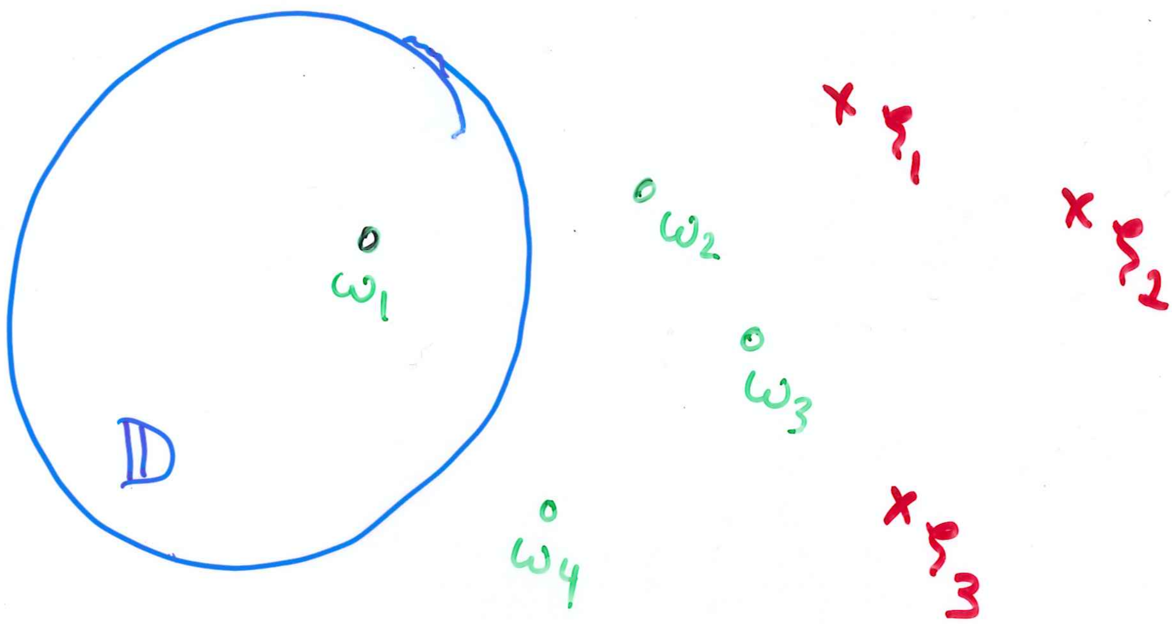
Abelian, or "multicut" case :

$$g(\xi,t) = f'(\xi,t) = \frac{\prod_1^m (\xi - \omega_k(t))}{\prod_1^n (\xi - \xi_j(t))}$$

$f(\xi,t)$ = rational + logarithmic terms

= Abelian integral

$|\xi_j| > 1$, <u>but allow</u> $\omega_k \in \mathbb{D}$



More difficult: allow $\omega_k \in \partial\mathbb{D}$

- Existence / uniqueness of sol. ?
- Dynamics of ω_k, ξ_j ?

↑

formulated by
Shraiman & Bensimon (1984)

- If $|\omega_k| > 1$ all k (local univalence):

$$PG \Leftrightarrow LK \,, \quad |\xi_j| \nearrow \infty \,,$$

ω_k more complicated (Yu-Lin Lin)
S. Tanveer, ...

- If $|\omega_k| < 1$ allowed:

$$PG \Leftarrow LK$$
$$\not\Rightarrow$$

Too many solutions

Good equation. Local existence and uniqueness
Problems if $\omega_k \in \partial\mathbb{D}$

$\boxed{\text{Abelian } f(\cdot, t) \text{ is a conserved class}}$

If $|\omega_k| = 1$:

Problems with both PG and LK

(unless $q(t) = 0$)

Assume $f(\xi, t)$ solves PG.

Then f solves LK if and only if

- $\dot{f}(\omega_k, t) = 0 \quad \forall \omega_k \in \mathbb{D}$

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or (equiv.):

- $\{f(\cdot, t)\}$ forms a <u>subordination chain</u>:

$$f(\xi, s) = f(\varphi(\xi, s, t), t)$$

for $s < t$, $\varphi(\cdot, s, t) : \mathbb{D} \hookrightarrow \mathbb{D}$

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or (equiv.):

- $\{f(\cdot, t)\}$ lift to a Riemann surface:

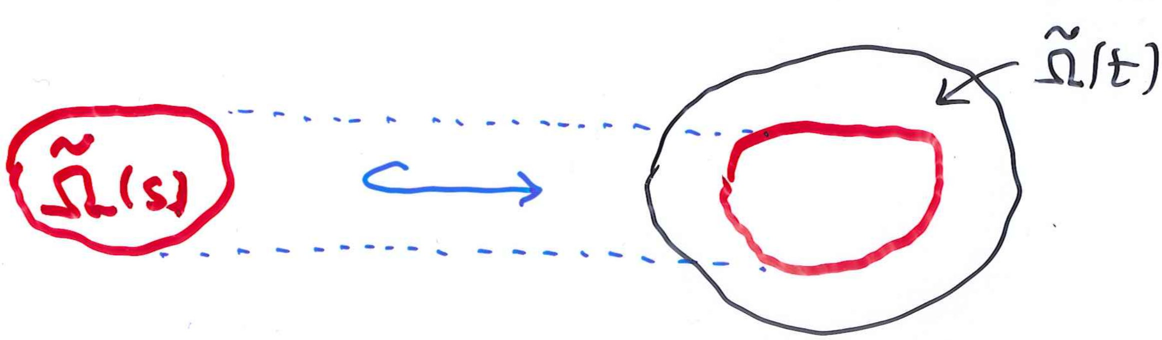
$$\begin{tikzcd}
 & \tilde{\Omega}(t) \arrow[r, phantom, "\subset"] & R \arrow[d, "p"] \\
\mathbb{D} \arrow[ur, "\tilde{f}(\cdot, t)", "\text{univ.}"'] \arrow[r, "f(\cdot, t)"'] & \Omega(t) \arrow[r, phantom, "\subset"] & \mathbb{C}
\end{tikzcd}$$

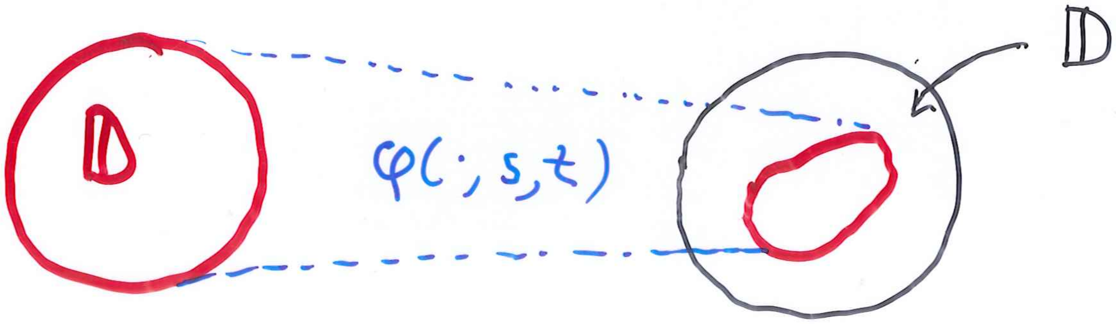
$$\begin{tikzcd}
 & \tilde{\Omega}(s) \arrow[r, hook] \arrow[d, "p"] & \tilde{\Omega}(t) \arrow[r, hook] \arrow[d, "p"] & R \arrow[d, "p"] \\
\mathbb{D} \arrow[r, "f(\cdot,s)"'] \arrow[ur, "\tilde{f}(\cdot,s)", "\text{univ.}"'] & \Omega(s) \arrow[r, hook] & \Omega(t) \arrow[r, hook] & \mathbb{C}
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathbb{D} \arrow[r, "\varphi(\cdot,s,t)", "1-1"'] \arrow[d, "f(\cdot,s)"] & \mathbb{D} \arrow[r] \arrow[d, "f(\cdot,t)"] & \boxed{?} \\
\mathbb{D} \arrow[r, "f(\cdot,s)"'] \arrow[ur, "id"] & \Omega(s) \arrow[r, hook] & \Omega(t) \arrow[r, hook] & \mathbb{C}
\end{tikzcd}$$

Define $R = \bigcup_{\text{all } t} \tilde{\Omega}(t)$

if $\{f(\cdot,t)\}$ is a subordination chain.





Main "theorem" (still shaky ...)

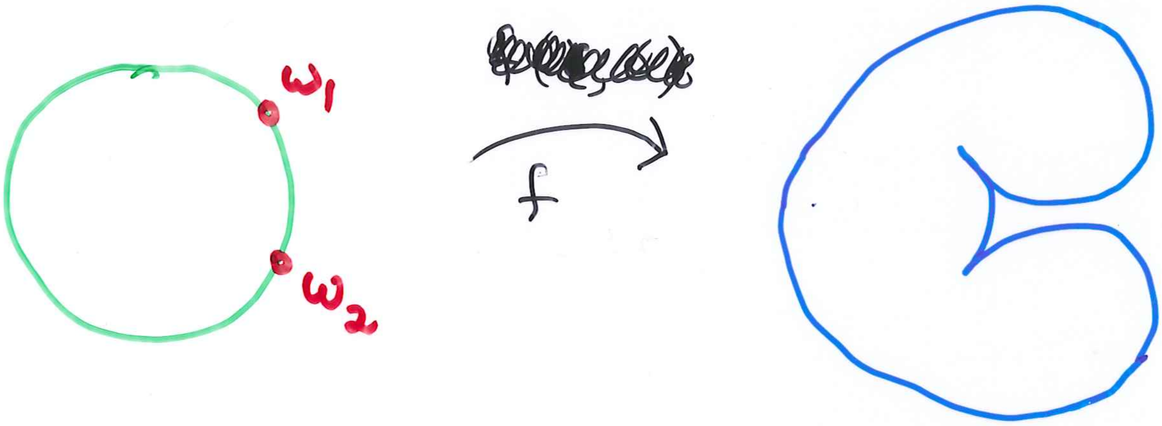
Given $f(\cdot,0) \in \mathcal{O}(\overline{\mathbb{D}})$ there exists $\{f(\cdot,t) \in \mathcal{O}(\mathbb{D}) : 0 \leq t < \infty\}$ solving LK in a weak sense.

$f(\cdot,0)$ Abelian $\Rightarrow$ $f(\cdot,t)$ Abelian $\forall t$ but the structure changes every time $\omega_k(t) \in \partial\mathbb{D}$ for some k.

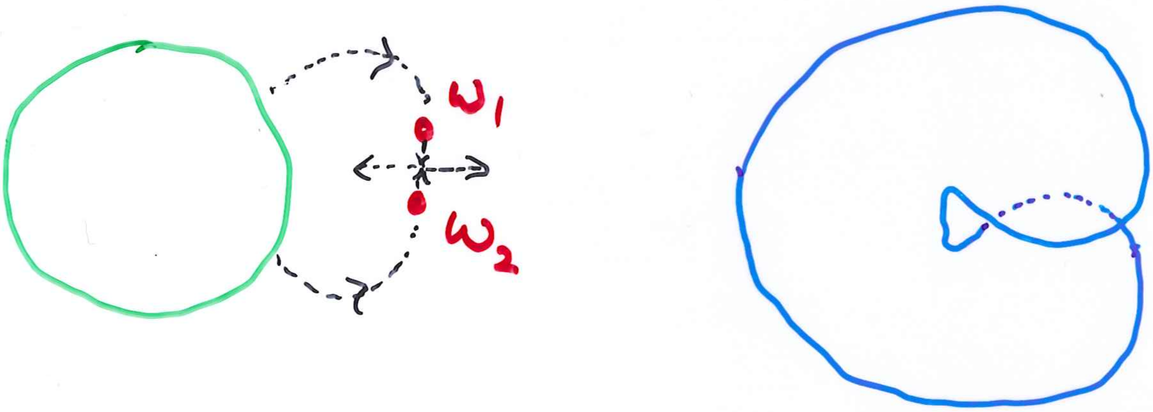
Remarks: Problems arise only when $\omega_k \in \partial\mathbb{D}$, and then the solution cannot be smooth.

Idea of proof: Lift the solution to the Riemann surface $R \xrightarrow{P} \mathbb{C}$ and use the variational inequality weak solution there.

Problem: R does not exist (in advance)



Later:

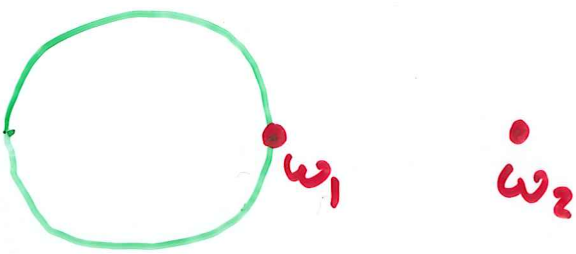


loss of [redacted] univalence

Bad prognosis.

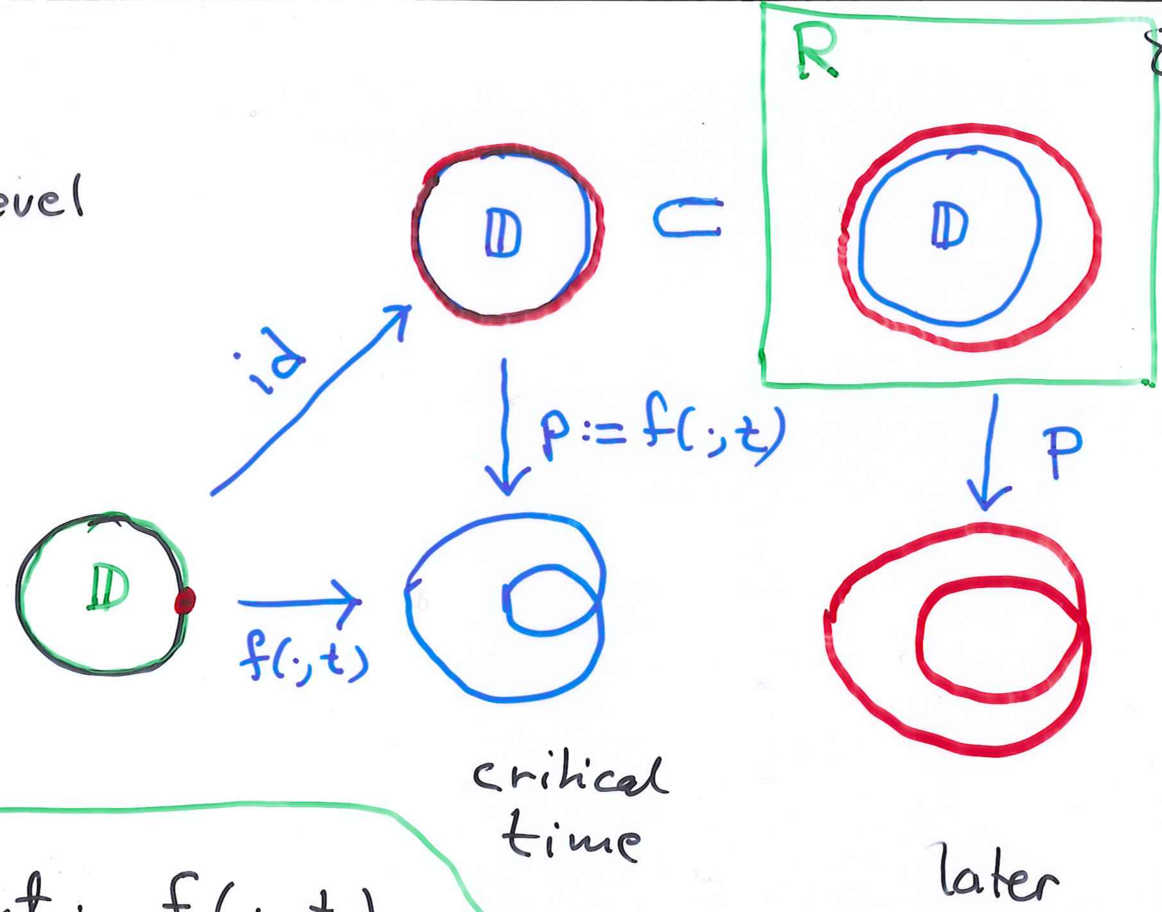
Time to lift to RS

Critical time:



serious problems

Riemann surface level



Important: $f(\cdot, t)$ is analytic in a larger domain, say R, so that $p := f(\cdot, t)$ can be used as covering map for a while.

Hele-Shaw flow on covering surface

$$\begin{array}{c} R \\ \downarrow p \\ \mathbb{C} \end{array}$$

$\tilde{z}$ coordinate on R.

$$dm_R(\tilde{z}) = |p'(\tilde{z})|^2 d\tilde{x}\, d\tilde{y} = \text{area form}$$

$$\frac{d}{dt} \int_{\tilde{\Omega}(t)} h \, dm_R = 2\pi q \cdot h(0) \quad \forall h \text{ harm.}$$

$(\Longleftrightarrow$ weighted Hele-Shaw flow $)$

PG: $\quad \mathrm{Re}\left[\dot{\tilde{f}} \cdot \overline{\zeta \tilde{f}'}\right] = \dfrac{q}{|p' \circ \tilde{f}|^2}$ on $\partial\mathbb{D}$

LK: $\quad \dot{\tilde{f}}(\zeta,t) = \zeta \tilde{f}'(\zeta,t) \cdot \underbrace{P(\zeta,t)}_{\text{defined in terms of } f}$ (in $\mathbb{D}$)

Weak formulation:

$$\int_{\tilde{\Omega}(t)} h\, dm_R - \int_{\tilde{\Omega}(s)} h\, dm_R \overset{\geq}{=} 2\pi\left(Q(t) - Q(s)\right) h(0)$$

$\forall h$ $\overset{\text{sub}}{\text{harmonic}}$ in $\tilde{\Omega}(t)$; $s \leq t$,

$Q(t) = \int_0^t q(\tau)\, d\tau$.

Given $\tilde{\Omega}(0) \subset R$, $\{\tilde{\Omega}(t) : 0 \leq t < \infty\}$ exists if just R is large enough.

But it may become multiply connected.

solution

To keep $\tilde{\Omega}(t)$ simply connected, by changing R every time simple connectivity is threatened, we need

Lemma (Sakai): Let $D(t)$, $0 \le t < \varepsilon$ be the weak solution satisfying

$$\int_{D(t)} h\,|p'|^2 dm = \int_{\mathbb{D}} h\,|p'|^2 dm + 2\pi Q(t)\, h(0)$$

Then $D(t)$ is simply connected for t small enough. ($D(0) = \mathbb{D}$).

Lemma: Radius of analyticity of $f(\cdot, t)$ increases with time.

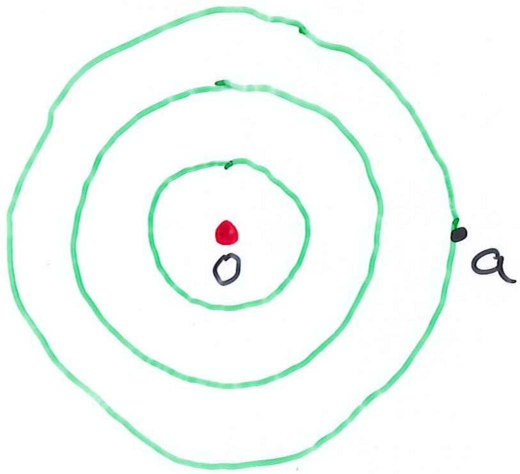
(True at least in locally univalent case. Hopefully true with suitable interpretation (analytic $\rightsquigarrow$ meromorphic) in general.)

Example: Take $a > 0$.

$-\pi a^2 < t < 0$:

$$\Omega(t) = \mathbb{D}\left(0, \sqrt{\frac{t+\pi a^2}{\pi}}\right)$$

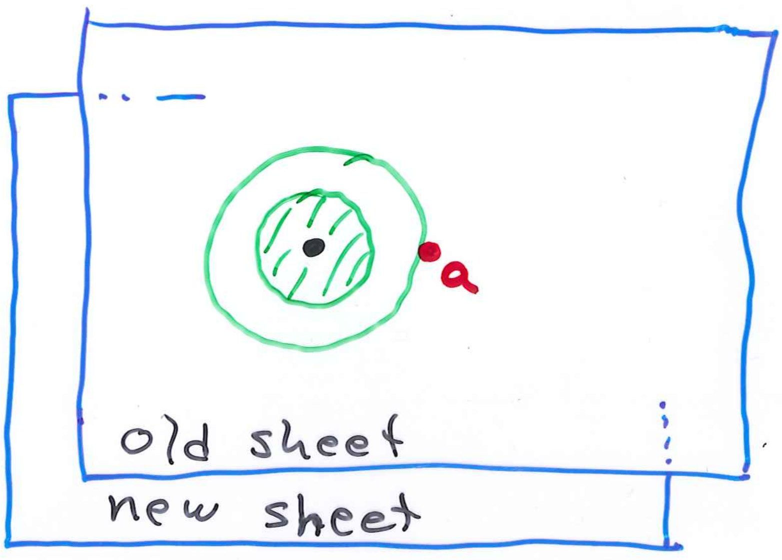
$$f(\xi, t) = \sqrt{\frac{t+\pi a^2}{\pi}} \cdot \xi$$



Not very exciting!

Start a "parallel universe" at the point a, to give the solution more options:

Riemann surface with branch point at a



$$R = \underbrace{(\mathbb{C} \setminus \{a\})}_{\text{old}} \cup \underbrace{(\mathbb{C} \setminus \{a\})}_{\text{new}} \cup \underbrace{\{a\}}_{\text{branch pt.}}$$

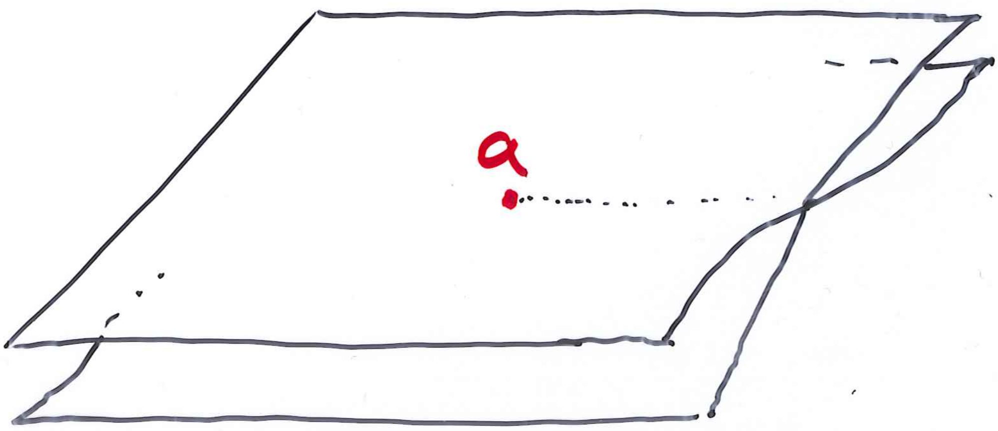
$p : R \to \mathbb{C}$ just identification

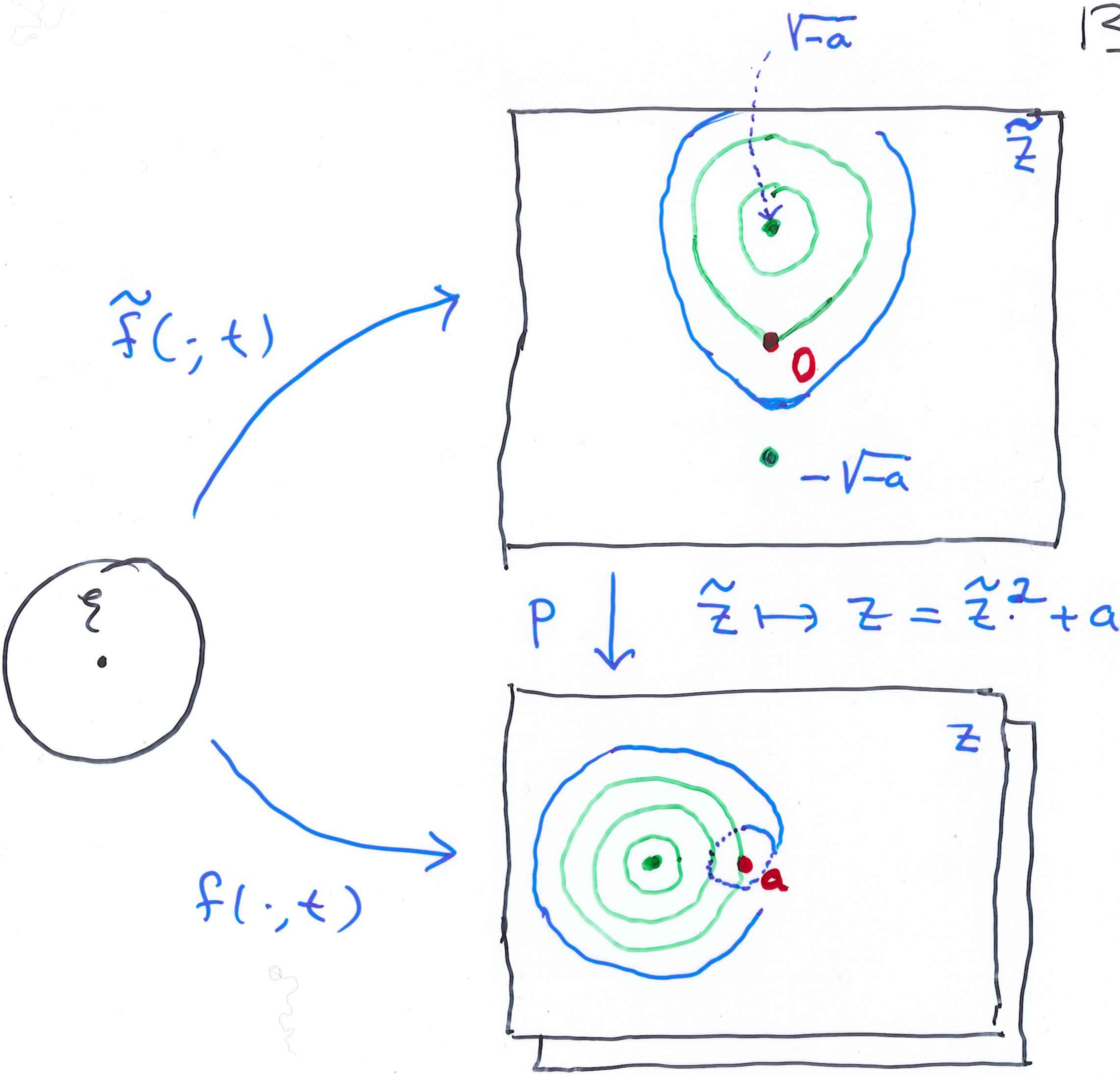
Coordinate on R : $\tilde{z}$ such that

$\tilde{z} = 0$ corresponds to a.

$$\begin{cases} p(\tilde{z}) = \tilde{z}^2 + a = z \\ \tilde{z} = \sqrt{z-a} \end{cases}$$

$\therefore$ R = classical Riemann surface of $\sqrt{z-a}$





For $t > 0$:

$$f(\xi,t) = a e^{3t}\, \xi \cdot \frac{\xi - 2e^{-t} + e^{-3t}}{\xi - e^{t}}$$

$$\tilde{f}(\xi,t) = \sqrt{a}\, e^{\frac{3t}{2}} \cdot \frac{\xi - e^{-t}}{\sqrt{\xi - e^{t}}}$$

The evolution on R is simply **weighted Hele-Shaw flow**:

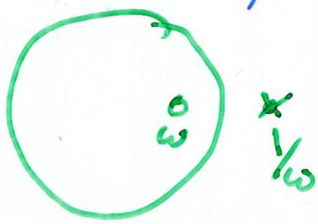
$$\frac{d}{dt}\int_{\tilde{\Omega}(t)} h(\tilde{z})\cdot \underbrace{4|\tilde{z}|^2 d\tilde{x}\, d\tilde{y}}_{dm_R(\tilde{z})} = 2\pi q(t)\, h(\sqrt{-a})$$

$a^2 e^{6t}(4-e^{-2t})$

Set $\omega = e^{-t} \in \mathbb{D}$. Then

$$g(\xi) = f'(\xi) = \frac{a}{\omega^3}\,\frac{(\xi-\omega)(\xi-\frac{2}{\omega}+\omega)}{(\xi-\frac{1}{\omega})^2},$$

$f(\omega) = a$,



$$\int_{\mathbb{D}} h|g|^2 dm = \text{const.}\ h(0) \qquad \forall h \text{ harm.}$$

$\because$ g **contractive zero divisor** in the sense of H. Hedenmalm (1991)

Also in M. Sakai (1988), paper on "quadrature Riemann surfaces".