Parametrization of the Loewner-Kufarev evolution in Sato's Grassmannian

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(joint work with Irina Markina)

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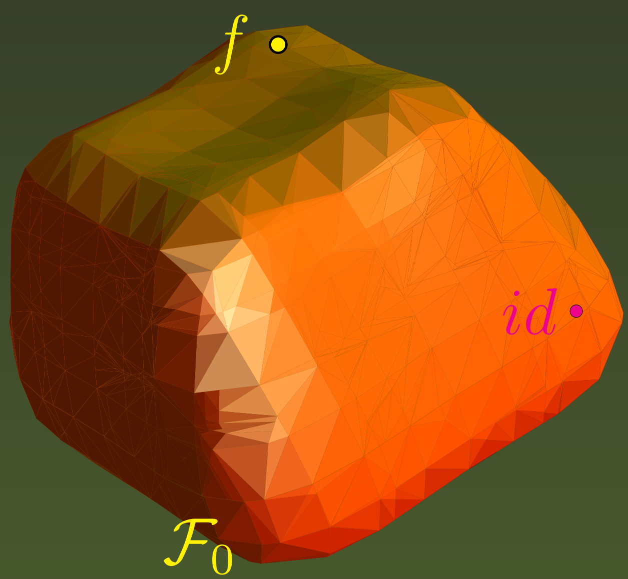
Idea



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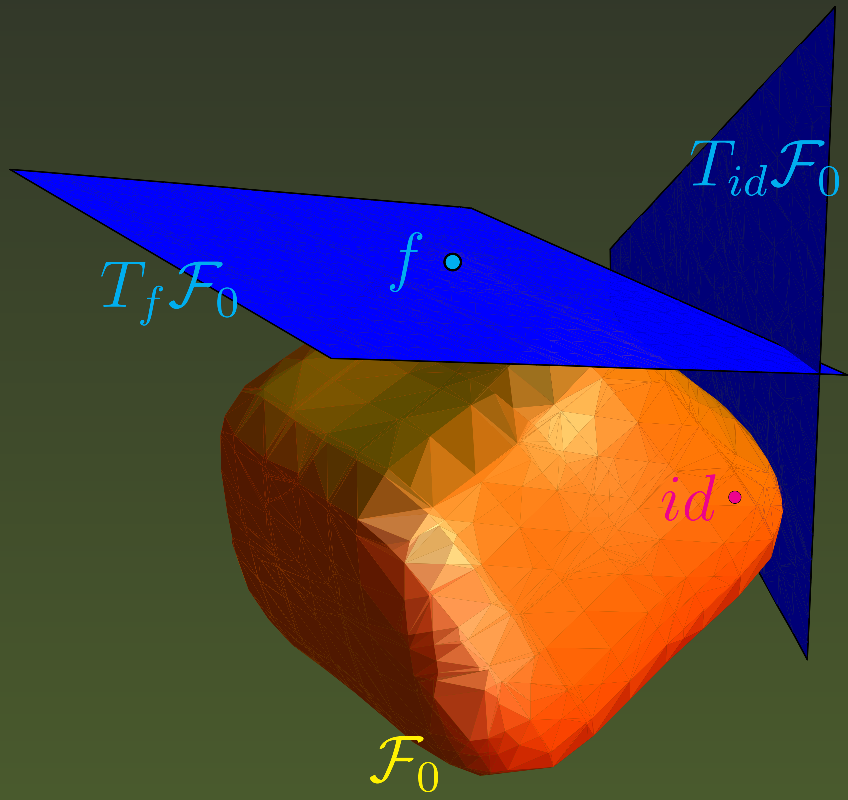

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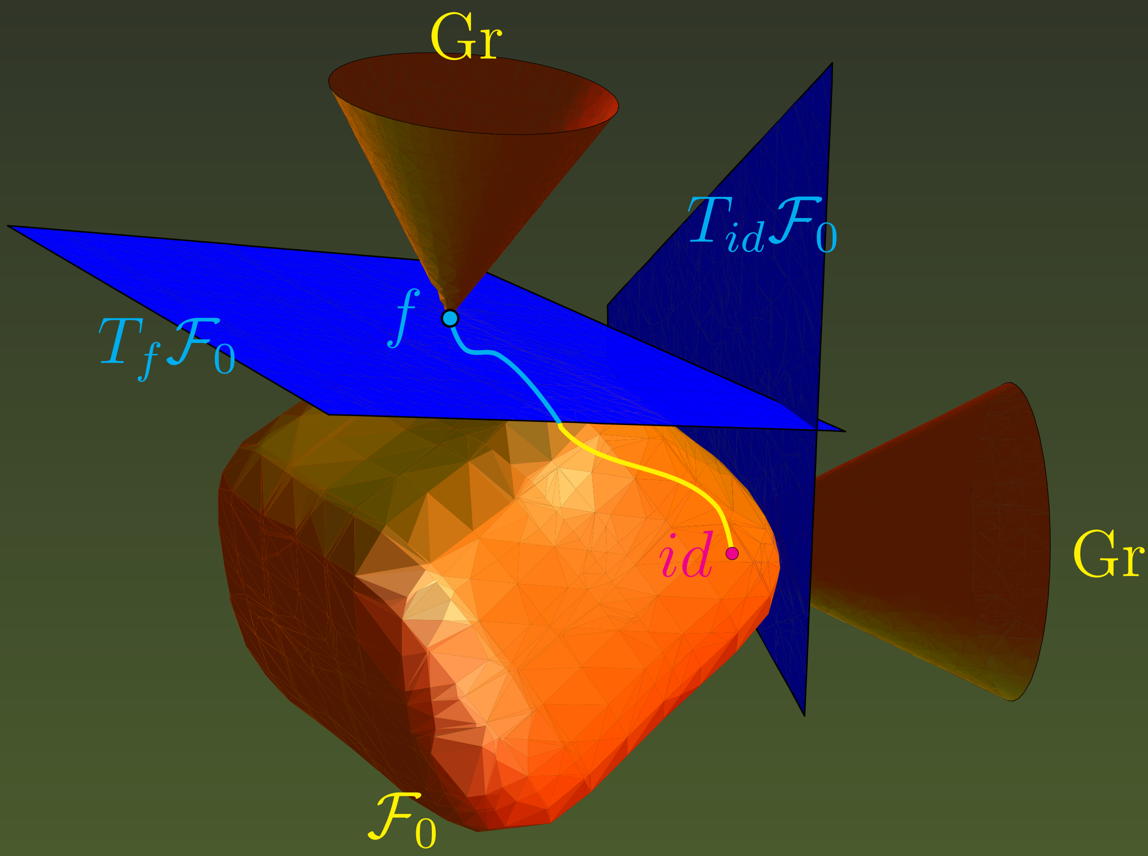

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- M. Sato, and Y. Sato, *Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold*.– Nonlinear Partial Differential Equations in Applied Science Tokyo, 1982, North-Holland Math. Stud. vol. 81, North-Holland, Amsterdam (1983), pp. 259–271.

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- Points of $\mathrm{Gr}(\mathcal{H})$ are closed linear subspaces V of $\mathcal{H}$ such that
  - orthogonal projection $\pi_+ : \quad V \longrightarrow \mathcal{H}_+$ is a Fredholm operator;
  - orthogonal projection $\pi_- : \quad V \longrightarrow \mathcal{H}_-$ is a Hilbert-Schmidt operator;

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Reminder:

- Fredholm: kernel and cokernel of π_+ are finite-dimensional;
- Hilbert-Schmidt: the norm $\left(\sum_{e_j} \|\pi_-(e_j)\|^2\right)^{1/2}$ is finite for some orthonormal basis $\{e_j\}$ in V.

Equivalent definition

- $V \in \mathrm{Gr}(\mathcal{H}) \Longleftrightarrow \exists$ bounded linear operator $\omega : \ \mathcal{H}_+ \longrightarrow \mathcal{H}$ such that
  - $\omega(\mathcal{H}_+) = V$;
  - $\pi_+ \circ \omega : \ \mathcal{H}_+ \longrightarrow \mathcal{H}_+$ is a Fredholm operator;
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- One-to-one correspondence between points of $\mathrm{Gr}(\mathcal{H})$ and Hilbert-Schmidt operators $\mathcal{L}^2(\mathcal{H}_+ \to \mathcal{H}_-)$.

Neighbourhoods

- $U_V \subset \mathrm{Gr}(\mathcal{H})$ consists of points V_T such that
  - $T \in \mathcal{L}^2(V \to V^\perp)$;
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- $\mathrm{Gr}(\mathcal{H})$ gets the structure of a smooth Hilbert manifold.

Smooth Grassmannian

- Hilbert space $\mathcal{H} = L^2(S^1 \to \mathbb{C})$ with the basis z^k, $k \in \mathbb{Z}$;
- Polarization $\mathcal{H}_+ = \mathrm{span}_{\mathbb{C}}\{z, z^2, \dots\}$;
- Index system $\mathbb{S}$ such that $\mathbb{S} \setminus \mathbb{N}$ and $\mathbb{N} \setminus \mathbb{S}$ are finite;
- $\mathcal{H}_{\mathbb{S}} = \mathrm{span}_{\mathbb{C}}\{z^k,\ k \in \mathbb{S}\}$.
- Any $V \in \mathrm{Gr}(\mathcal{H})$ is isomorphic to $\mathcal{H}_{\mathbb{S}}$ for some $\mathbb{S}$.

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- $\mathcal{H}$ we replace by $\mathcal{H} \cap C^\infty(S^1 \to \mathbb{C})$;
  - $\pi_+ : \quad V \to \mathcal{H}_+ \cap C^\infty$ is a Fredholm operator;
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We obtain a dense submanifold Gr_∞ of $\mathrm{Gr}(\mathcal{H})$.

Virtual dimension

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For example:

- $V \ni$ span$\{z^{-n}, z^{-n+1}, \ldots\} \ni \psi = \sum_{k=-n}^{+\infty} \psi_k z^k \Longrightarrow$ virt.dim$(V) = n + 1$.

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- Its central extension $Vir = \text{Diff } S^1 \oplus \mathbb{R}$– Virasoro-Bott group;
- Its quotient Diff S^1/S^1– Kirillov's manifold;
- Groups Diff S^1 and Vir and the homogeneous manifold Diff S^1/S^1 are modeled on Fréchet spaces.

... and their infinitesimal representations.

Virasoro group and algebra

- The Lie algebra $\mathfrak{diff}\, S^1$ for the group $\mathrm{Diff}\, S^1$ is isomorphic to the Lie algebra $\mathrm{Vect}\, S^1$ of smooth vector fields $v = v(\theta)\frac{d}{d\theta}$ on S^1.

Virasoro group and algebra

- $Vir = \text{Diff}\, S^1 \oplus \mathbb{R} \longrightarrow \mathfrak{vir}$;
- $\text{Diff}\, S^1 \longrightarrow \text{Vect}\, S^1$;
- $\text{Diff}\, S^1 / S^1 \longrightarrow \text{Vect}\,_0 S^1$.

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Complexifcation:

- $(Vir, T^{(1,0)}_{\mathfrak{vir}})$ $\qquad T^{(1,0)}_{\mathfrak{vir}} \oplus T^{(0,1)}_{\mathfrak{vir}} = \mathfrak{vir} \otimes \mathbb{C}$;
- $(\text{Diff } S^1, H^{(1,0)})$ $\qquad H^{(1,0)} \oplus H^{(0,1)} = \text{corank}_1(\text{Vect } S^1 \otimes \mathbb{C})$;
- $(\text{Diff } S^1/S^1, T^{(1,0)})$ $\qquad T^{(1,0)} \oplus T^{(0,1)} = \text{Vect }_0 S^1 \otimes \mathbb{C}$;

Relation to analytic functions

- $\mathcal{F}$ smooth univalent functions in $\mathbb{D}$,
  $f(z) = z(a_0 + a_1 z + \dots);$
- $\mathcal{F}_1 \subset \mathcal{F}$ conformal radius $f(\mathbb{D})$ is 1;
- $\mathcal{F}_0 = \mathcal{F} \bigcap \mathbf{S}$, where **S** normalized univalent functions in $\mathbb{D}$,
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Mappings:

$$(Vir, T^{(1,0)}_{\mathfrak{vir}}) \xrightarrow{Hol} \mathcal{F};$$

$$(\text{Diff } S^1, H^{(1,0)}) \xrightarrow{C-R} \mathcal{F}_1;$$

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- $(\text{Diff } S^1, H^{(1,0)}) \xrightarrow{C-R} \mathcal{F}_1$ pseudoconvex hypersurface in $\mathcal{F}$.
- Lie hull $(H^{(1,0)}, H^{(0,1)}) \not\subset \{H^{(1,0)} \oplus H^{(0,1)}\}$.

Principal bundles

- Principal S^1 bundle:

$$\mathrm{U}(1) \simeq S^1 \longrightarrow \mathrm{Diff}\, S^1 \longrightarrow \mathrm{Diff}\, S^1/S^1;$$

- Trivial principal $\mathbb{C}^*$ bundle ($\mathbb{C}^* = \mathbb{C} \setminus \{0\}$):

$$\mathbb{C}^* \longrightarrow Vir_{\mathbb{C}} \simeq \mathcal{F} \longrightarrow \mathrm{Diff}\,_{\mathbb{C}} S^1/S^1 \simeq \mathcal{F}_0.$$

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- $[v_n, v_m] = (m-n)v_{n+m}$;

- (v_n, a), $(v_m, b) \in \mathfrak{vir}_{\mathbb{C}}$;
  $a, b, c \in \mathbb{C}$,
  c is a constant in the co-cycle,

$$[(v_n, a), (v_m, b)]_{\mathfrak{vir}} = \Big((m-n)v_{n+m}, \frac{c}{12}n(n^2-1)\delta_{n,-m}\Big).$$

Univalent Functions

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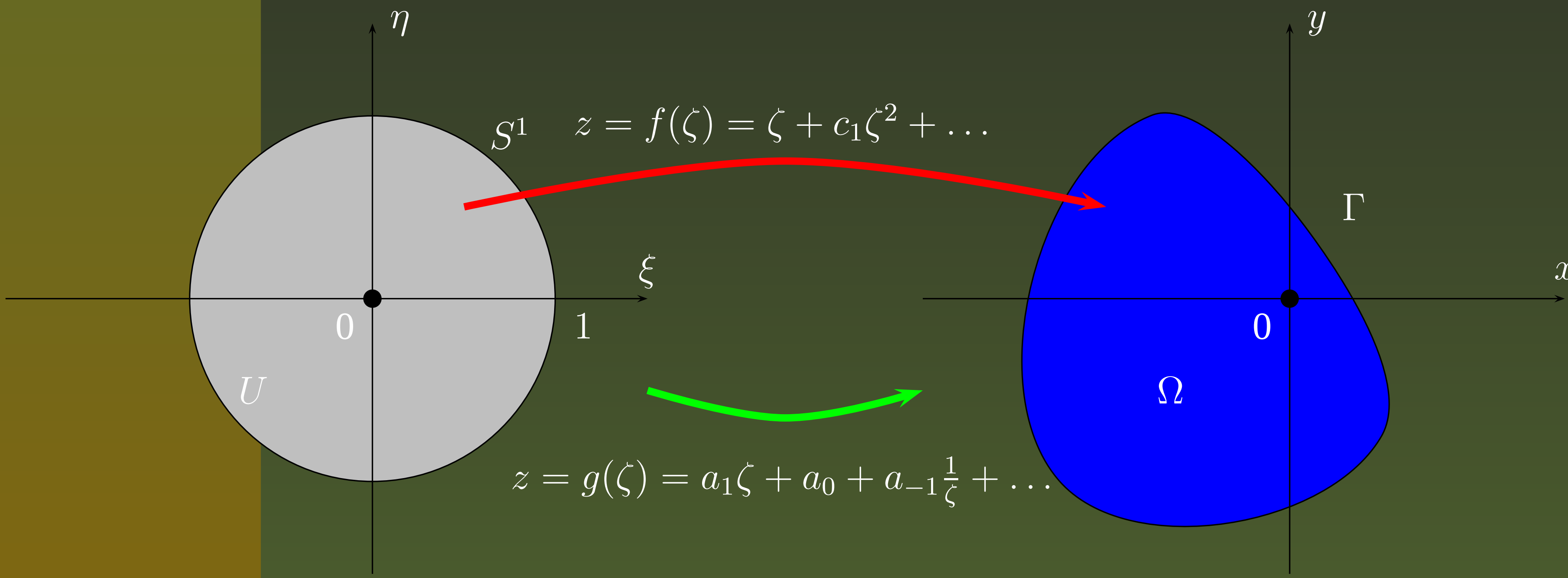

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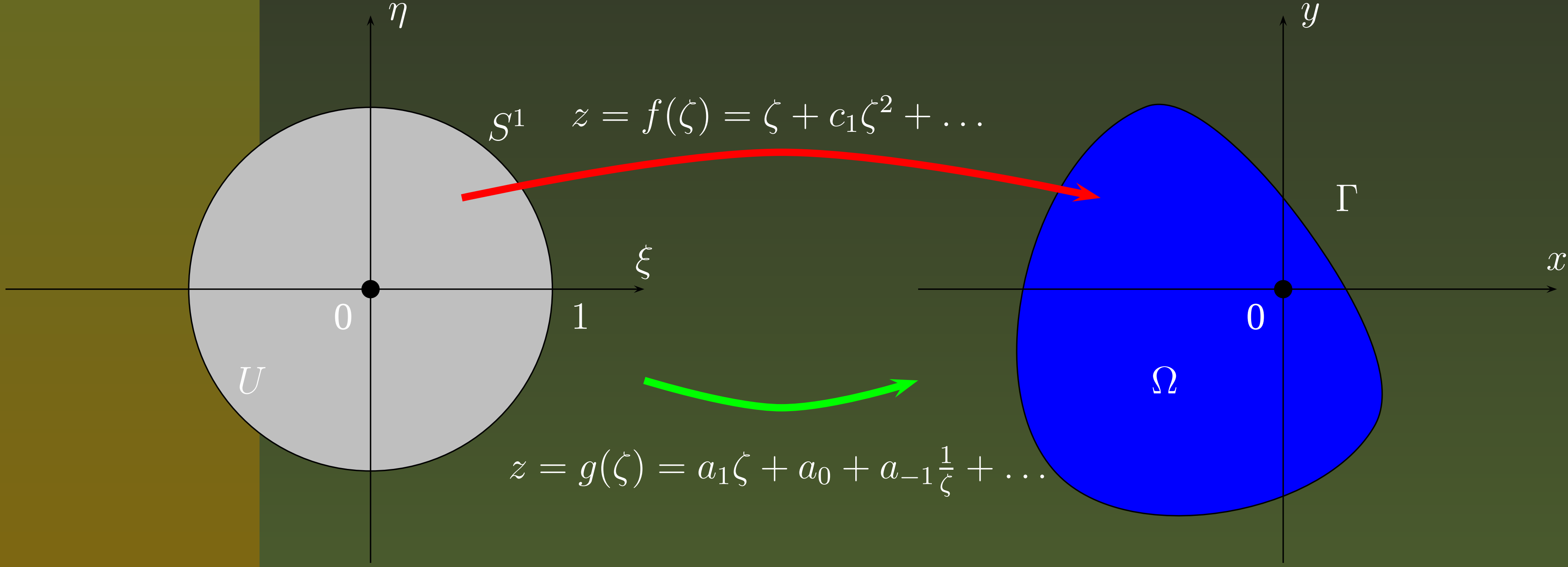

- $\gamma = f^{-1} \circ g|_{S^1} \in \text{Diff } S^1/S^1, \quad f \in \mathcal{F}_0 \leftrightarrows \gamma \in \text{Diff } S^1/S^1.$

Kirillov's vector fields

Schaeffer and Spencer linear operator

$$\frac{f^2(\zeta)}{2\pi}\int_{S^1}\left(\frac{wf'(w)}{f(w)}\right)^2\frac{v(w)dw}{w(f(w)-f(z))},$$

maps Vect $_0S^1 \longrightarrow T_f\mathcal{F}_0$ and

Vect $_0S^1 \otimes \mathbb{C} = T^{(1,0)} \oplus T^{(0,1)} \longrightarrow T_f\mathcal{F}_0 \otimes \mathbb{C}$.

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- Taking Fourier basis $v_k = -iz^k$, $k = 1, 2, \dots$ for $T^{(1,0)}$, we obtain

$$L_k[f](z) = z^{k+1}f'(z).$$

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Vect $_0S^1\otimes\mathbb{C}=T^{(1,0)}\oplus T^{(0,1)}\longrightarrow T_f\mathcal{F}_0\otimes\mathbb{C}$.

- Taking $v_{-k}=-iz^{-k}$, $k=1,2,\ldots$ for $T^{(0,1)}$, we obtain

$$L_{-k}[f](\zeta)=\text{very difficult expressions.}$$

Kirillov's vector fields

- Virasoro commutation relation

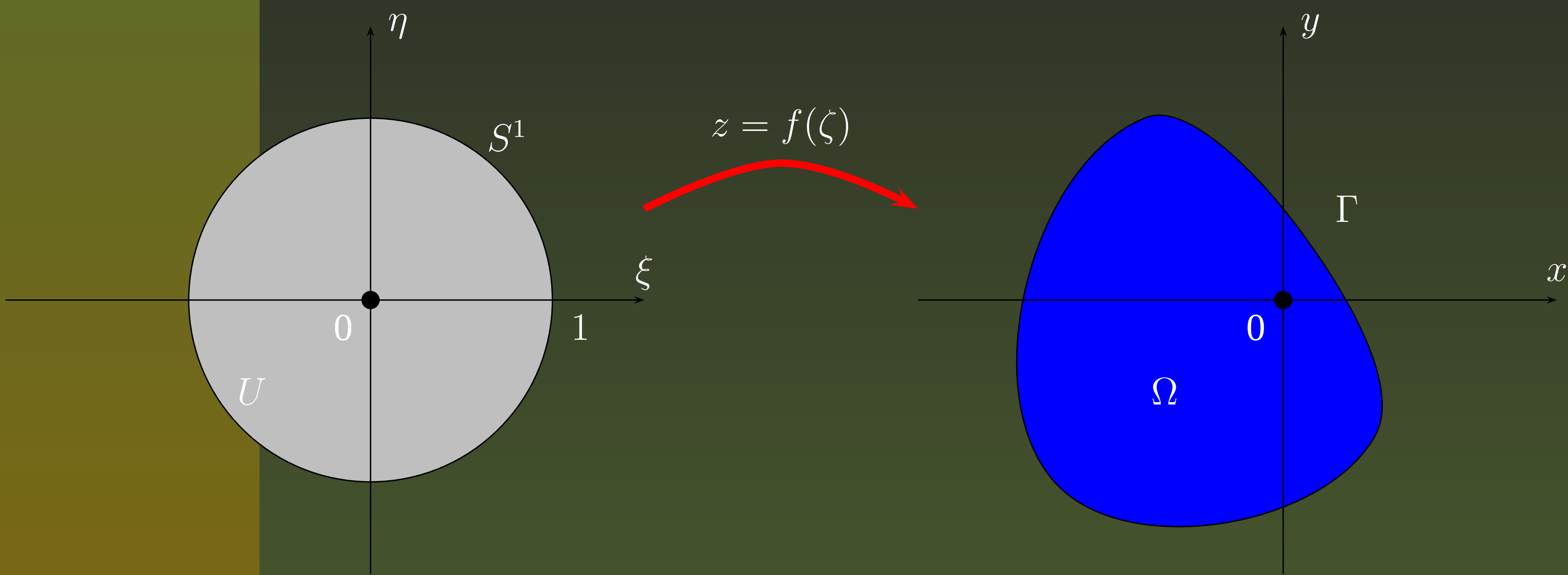
$$[L_m, L_n]_{\mathfrak{vir}} = (m-n)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{n,-m},$$

$c \in \mathbb{C}$. $L_0[f](z) = zf'(z) - f(z)$ corresponds to rotation.

- In affine coordinates we get Kirillov's operators for $n = 1, 2, \dots :$

$$L_n = \partial_n + \sum_{k=1}^{\infty}(k+1)c_k\partial_{n+k}, \quad \partial_k = \partial/\partial c_k,$$

Univalent Functions



Class **S**.

Löwner-Kufarev Representation

- Any univalent function $f : U \to \Omega$, $f(z) = z + c_1 z^2 + \dots$ (from the class **S**) can be represented as the limit

$$f(z) = \lim_{t\to\infty} e^t w(z,t).$$

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- The function $\zeta = w(z, t)$,

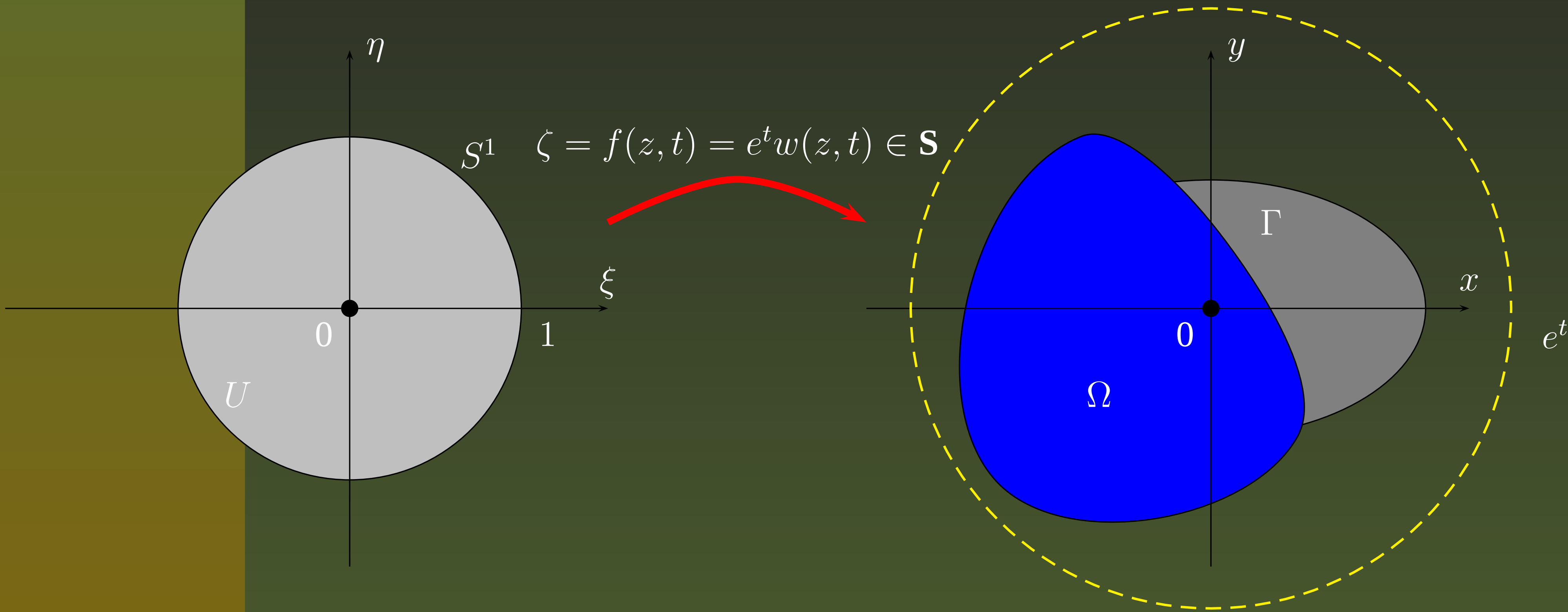
$$w(z,t) = e^{-t} z \left(1 + \sum_{n=1}^{\infty} c_n(t) z^n \right),$$

satisfies

$$\frac{dw}{dt} = -wp(w, t),$$

with the initial condition $w(z, 0) = z$.

Non-linear contour dynamics



We shall consider functions $p(z,t)$ smooth on S^1 and integrable $\Longrightarrow$ $f \in \mathcal{F}_0$.

Hamiltonian formulation

- Consider the Hamiltonian on the cotangent bundle $T^*\mathcal{F}_0$:

$$H(f,\bar{\psi}) = \int\limits_{z\in S^1} f(z,t)(1-p(e^{-t}f(z,t),t))\bar{\psi}(z,t)\frac{dz}{iz},$$

on the unit circle $z \in S^1$, where $\psi(z,t)$ is from $L^2(S^1 \to \mathbb{C})$,

$$\psi(z,t) = \sum_{n\in\mathbb{Z}} \psi_k z^k.$$

Hamiltonian system

- The Hamiltonian system becomes

$$\frac{df(z,t)}{dt} = f(1 - p(e^{-t}f,t)) = \frac{\delta H}{\delta \overline{\psi}} = \{f, H\}$$

for the position coordinates and

$$\frac{d\bar{\psi}}{dt} = -(1-p(e^{-t}f,t)-e^{-t}fp'(e^{-t}f,t))\bar{\psi} = \frac{-\delta H}{\delta f} = \{\overline{\psi}, H\},$$

for the momenta, where $\frac{\delta}{\delta f}$ and $\frac{\delta}{\delta \overline{\psi}}$ are the variational derivatives.

Poisson structure

- The Poisson structure on the space $T^*\mathcal{F}_0$ with coordinates $(f, \bar{\psi})$ is given by the canonical brackets

$$\{P, Q\} = \frac{\delta P}{\delta f}\frac{\delta Q}{\delta \bar{\psi}} - \frac{\delta P}{\delta \bar{\psi}}\frac{\delta Q}{\delta f},$$

or in affine coordinate form (only ψ_n for $n \geq 1$ are independent)

$$\{p, q\} = \sum_{n=1}^{\infty} \frac{\partial p}{\partial c_n}\frac{\partial q}{\partial \bar{\psi}_n} - \frac{\partial p}{\partial \bar{\psi}_n}\frac{\partial q}{\partial c_n}.$$

Generating function

- Set up the function $\mathcal{G}(z) := \bar{f}'(z,t)\psi(z,t) \in L^2(S^1 \to \mathbb{C})$.
- Let $(\mathcal{G}(z))_{\leq 0}$ mean the 'negative' and $(\mathcal{G}(z))_{>0}$ 'positive' part of the Laurent series for $\mathcal{G}(z)$,

$$(\mathcal{G}(z))_{>0} = (\psi_1+2\bar{c}_1\psi_2+3\bar{c}_2\psi_3+\dots)z+(\psi_2+2\bar{c}_1\psi_3+\dots)z^2+\dots$$

$$= \sum_{k=1}^{\infty} \bar{\mathcal{G}}_k z^k.$$

Proposition. The functions $\mathcal{G}(z)$, $(\mathcal{G}(z))_{\leq 0}$ and $(\mathcal{G}(z))_{>0}$ are time-independent for all $z \in S^1$.

Invertibility

$$\begin{pmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \\ \mathcal{G}_3 \\ \dots \end{pmatrix} = \begin{pmatrix} 1 & 2c_1 & \dots & nc_{n-1} & \dots \\ 0 & 1 & \dots & (n-1)c_{n-2} & \dots \\ 0 & 0 & \dots & (n-2)c_{n-3} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \\ \dots \end{pmatrix};$$

Proposition.

- $\mathcal{G} = C\bar{\Psi}$ and $\exists\ \bar{\Psi} = C^{-1}\mathcal{G}$;
- $\bar{\psi}_k = \bar{\psi}_k(\mathcal{G}_k, \mathcal{G}_{k+1}, \dots)$.

Kirillov's vector fields

- $\{\mathcal{G}_m, \mathcal{G}_n\} = (n-m)\mathcal{G}_{n+m}$ for $n, m \geq 1$, with respect to our Poisson structurem.
- From co-vectors to vectors $\bar{\psi}_k \to \dfrac{\partial}{\partial c_k} = \partial_k$,

$$\mathcal{G}_n \to L_n = \partial_n + \sum_{k=1}^{\infty}(k+1)c_k\partial_{n+k}.$$

- L_n, $n = 1, 2, \ldots$ are the holomorphic Virasoro generators. In their covariant form, L_n are conserved by the Löwner-Kufarev evolution.

Graphs in Grassmannian Gr_∞

- Underlying Hilbert space

$$\mathcal{H} = L^2(S^1 \to \mathbb{C}) \cap C^\infty(S^1 \to \mathbb{C});$$

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- Polarization:

$$\begin{aligned}
\mathcal{H}_+ &= \mathrm{span}_\mathbb{C}\{z, z^2, z^3, \dots\} \cap L^2 \cap C^\infty, \\
\mathcal{H}_- &= \mathrm{span}_\mathbb{C}\{1, z^{-1}, z^{-2}, \dots\} \cap L^2 \cap C^\infty.
\end{aligned}$$

Graphs in Grassmannian Gr_∞

- Consider a neighbourhood $U_{\mathcal{H}_+}$ of the element $\mathcal{H}_+$;
- Construct a hierarchy of Hilbert-Schmidt operators $T_{-n}: \ \mathcal{H}_+ \to \mathcal{H}_-$:

$$T_{-n}(L_1, L_2, \dots, L_k, \dots) = \begin{cases} L_0(L_1, L_2, \dots, L_k, \dots) \\ L_{-1}(L_1, L_2, \dots, L_k, \dots) \\ \dots \\ L_{-n}(L_1, L_2, \dots, L_k, \dots) \end{cases}$$

Graphs in Grassmannian Gr_∞

- Define $\bar{\psi}_0^*(L_1,\dots) = -\sum_{n=1}^{\infty} c_k \bar{\psi}_k(L_1,\dots)$, and

$$L_0 = \mathcal{G}_0 - (\bar{\psi}_0 - \bar{\psi}_0^*)$$

- L_0 acts on the class $\mathcal{F}_0$ by $L_0[f](z) = zf'(z) - f(z)$.

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- Next define $L_{-1} = \mathcal{G}_{-1} - (\bar\psi_{-1} - \bar\psi^*_{-1}) - 2c_1(\bar\psi_0 - \bar\psi_0^*)$, where $\bar\psi^*_{-1} = 0$. Then,

$$L_{-1}[f](z) = f'(z) - 2c_1 f(z) - 1$$

- Finally,

$$L_{-2} = \mathcal{G}_{-2} - (\bar\psi_{-2} - \bar\psi^*_{-2}) - 2c_1(\bar\psi_{-1} - \bar\psi^*_{-1}) - 3c_2(\bar\psi_0 - \bar\psi_0^*)$$

Graphs in Grassmannian Gr_∞

First 3 antiholomorphic Virasoro generators:

- $L_0[f](z) = zf'(z) - f(z)$;
- $L_{-1}[f](z) = f'(z) - 2c_1 f(z) - 1$;
- $L_{-2}[f](z) = \frac{f'(z)}{z} - \frac{1}{f(z)} - 3c_1 + (c_1^2 - 4c_2)f(z)$.
- Important fact:

$$L_0 = c_1\bar{\psi}_1 + 2c_2\bar{\psi}_2 + \dots,$$

$$L_{-1} = (3c_2 - 2c_1^2)\bar{\psi}_1 + \dots,$$

$$L_{-2} = (5c_3 - 6c_1c_2 + 2c_1^3)\bar{\psi}_1 + \dots,$$

are co-vectors.

Graphs in Grassmannian Gr_∞

- Other co-vectors we construct by our Poisson brackets as

$$L_{-n} = \frac{1}{n-2}\{L_{-n+1}, L_{-1}\}.$$

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Graphs in Grassmannian Gr_∞

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Principal bundle over $\mathcal{F}_0$

- The base of Gr_∞ is the Hilbert space $L^2(S^1 \to \mathbb{C}) \cap C^\infty$;
- The functions $\mathcal{G}(z) = \sum_{k\in\mathbb{Z}} \bar{\mathcal{G}}_k z^k$ at a point $f \in \mathcal{F}_0$ are completely defined by their values at $f = id$.

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- Isomorphism between fibers is given by the Hamiltonian flow $\Longrightarrow$ principal bundle $\mathfrak{E} = (\mathcal{F}_0, \mathfrak{W})$ over $\mathcal{F}_0$ with fiber $\mathfrak{W}$.

Conclusions

- We have constructed a bundle $\mathfrak{E} = (\mathcal{F}_0, \mathfrak{W})$ over the base manifold $\mathcal{F}_0$ of smooth univalent functions.
- The fiber $\mathfrak{W}$ consists of special points of Sato's smooth Grassmannian Gr_∞;

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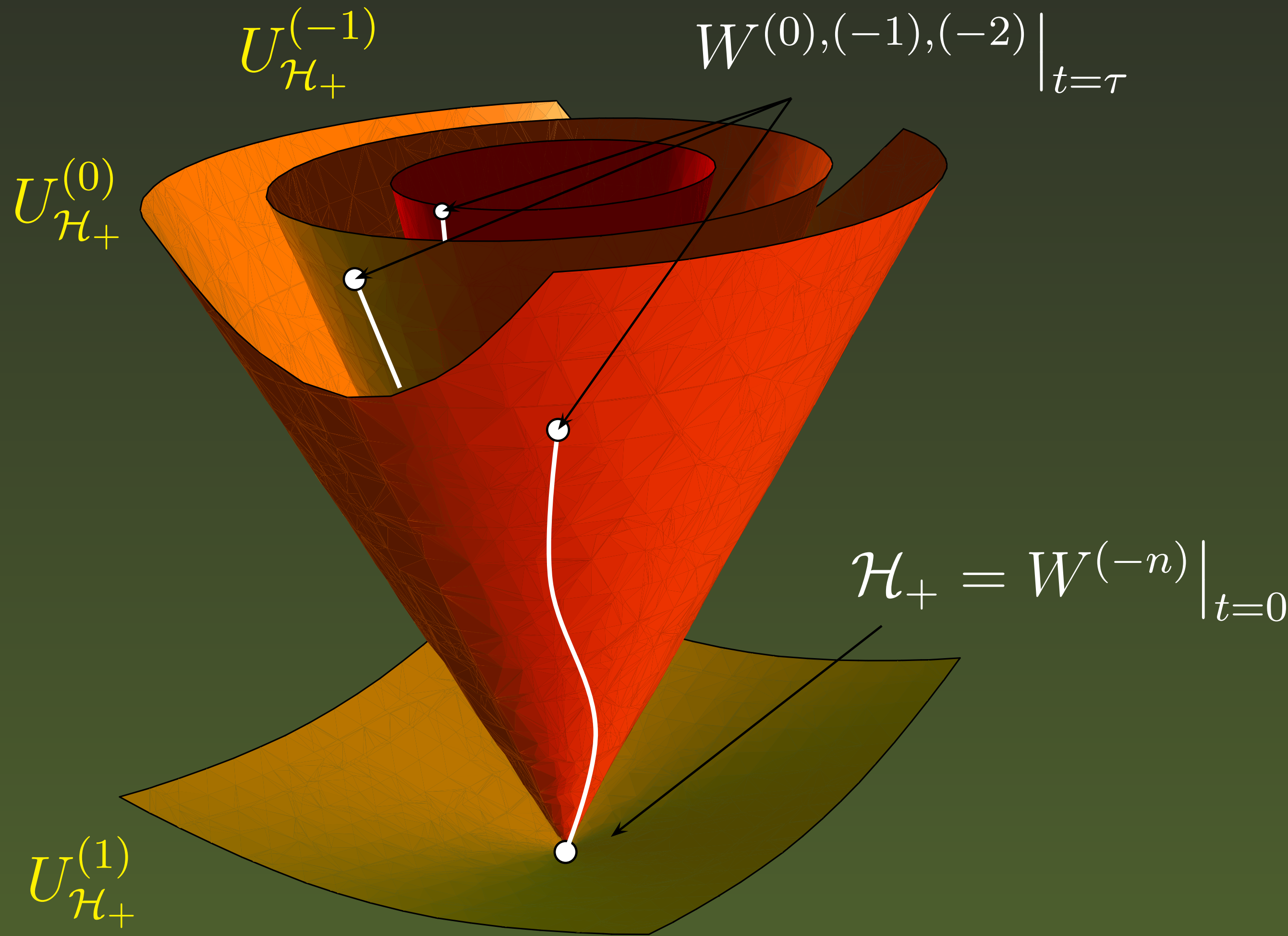
- The fiber $\mathfrak{W}$ consists of special points of Sato's smooth Grassmannian Gr_∞;
- It is a principle bundle. Isomorphism between fibers is given by the Hamiltonian flow.
- The Löwner-Kufarev evolution in $\mathcal{F}_0$ traces a curve in the principal bundle $\mathfrak{E}$, which is projected to a curve in each connected component $U_{\mathcal{H}_+}^{(-n)}$ of the neighbourhood $U_{\mathcal{H}_+}$ of the point $\mathcal{H}_+ \in \mathrm{Gr}_\infty$;

Conclusions

- The $(-n)$-th component $U_{\mathcal{H}_+}^{(-n)}$ is defined by its virtual dimension Virt.dim($U_{\mathcal{H}_+}^{(-n)}$)$= n+1$.
- The component $U_{\mathcal{H}_+}^{(-n)}$ contains a point of Gr_∞ defined by the graph $W^{(-n)} = (Id + T_{-n})H_+ \in \mathrm{Gr}_\infty$.

Löwner-Kufarev traces in Gr_∞



$\mathcal{H}_+ \in U^{(1)}_{\mathcal{H}_+}$; ○—○ Löwner-Kufarev traces

The End