The Convex Hull of a Space Curve

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Reference

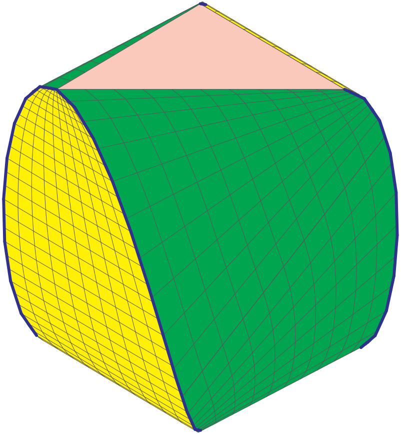
-, Bernd Sturmfels: *On the convex hull of a space curve*, math.AG/0912.2986

Consider the trigonometric space curve defined parametrically by

$$x = \cos(\theta)\,,\;\; y = \sin(2\theta)\,,\;\; z = \cos(3\theta). \tag{0.1}$$

This is an algebraic curve of degree 6 cut out by intersecting two surfaces of degree 2 and 3:

$$x^2 - y^2 - xz \;\; = \;\; z - 4x^3 + 3x \;\; = \;\; 0. \tag{0.2}$$



The convex hull of the curve $(\cos(\theta), \sin(2\theta), \cos(3\theta))$ has two triangles and two non-linear surfaces patches of degree 3 and 16 in its boundary.

The convex hull of our curve is the following projection of a 6-dimensional spectrahedron:

$$\text{Figure} = \left\{ (x,y,z) \in \mathbb{R}^3 \mid \exists\, u,v,w \in \mathbb{R} : \begin{pmatrix} 1 & x+ui & v+yi & z+wi \\ x-ui & 1 & x+ui & v+yi \\ v-yi & x-vi & 1 & x+ui \\ z-wi & v-yi & x-ui & 1 \end{pmatrix} \succeq 0 \right\}.$$

Here "$\succeq 0$" means that this Hermitian 4×4-matrix is positive semidefinite.
The boundary surface of the convex hull is not easily derived from this representation.

The yellow surface has degree 3 and is defined by

$$z - 4x^3 + 3x \ = \ 0. \tag{0.3}$$

The green surface has degree 16 and its defining polynomial is ....

$$1024x^{16} - 12032x^{14}y^2 + 52240x^{12}y^4 - 96960x^{10}y^6 + 56160x^8y^8 + 19008x^6y^{10} + 1296x^4y^{12} + 6144x^{15}z - 14080x^{1}$$
$$-72000x^{11}y^4z + 149440x^9y^6z + 79680x^7y^8z + 7488x^5y^{10}z + 15360x^{14}z^2 + 36352x^{1}2y^2z^2 + 151392x^{1}0y^4z^2 + 131264$$
$$+18016x^6y^8z^2 + 20480x^{1}3z^3 + 73216x^{1}1y^2z^3 + 105664x^9y^4z^3 + 23104x^7y^6z^3 + 15360x^{1}2z^4 + 41216x^{1}0y^2z^4 + 16656$$
$$+6144x^{11}z^5 + 6400x^9y^2z^5 + 1024x^{10}z^6 - 26048x^{14} - 135688x^{12}y^2 + 178752x^{10}y^4 + 124736x^8y^6 - 210368x^6y^8 + 79$$
$$+5184x^2y^{12} + 432y^{14} - 77888x^{13}z + 292400x^{11}y^2z + 10688x^9y^4z - 492608x^7y^6z - 67680x^5y^8z + 21456x^3y^{10}z + 25$$
$$-81600x^{12}z^2 - 65912x^{10}y^2z^2 - 464256x^8y^4z^2 - 192832x^6y^6z^2 + 31488x^4y^8z^2 + 6552x^2y^{10}z^2 - 40768x^{11}z^3 - 19440$$
$$-196224x^7y^4z^3 + 14912x^5y^6z^3 + 8992x^3y^8z^3 - 20800x^{10}z^4 - 84088x^8y^2z^4 - 7360x^6y^4z^4 + 7168x^4y^6z^4 - 12480$$
$$-9680x^7y^2z^5 + 3264x^5y^4z^5 - 2624x^8z^6 + 760x^6y^2z^6 + 64x^7z^7 + 189649x^{12} + 104700x^{10}y^2 - 568266x^8y^4 + 26882$$
$$+118497x^4y^8 - 42984x^2y^{10} - 432y^{12} + 62344x^{11}z - 592996x^9y^2z + 421980x^7y^4z + 377780x^5y^6z - 79748x^3y^8z - 18$$
$$+104620x^{10}z^2 + 56876x^8y^2z^2 + 480890x^6y^4z^2 - 12440x^4y^6z^2 - 51354x^2y^8z^2 - 936y^{10}z^2 + 35096x^9z^3 + 181132x^7$$
$$+73800x^5y^4z^3 - 52792x^3y^6z^3 - 3780xy^8z^3 - 6730x^8z^4 + 52596x^6y^2z^4 - 19062x^4y^4z^4 - 5884x^2y^6z^4 + y^8z^4 + 600$$
$$+2516x^5y^2z^5 - 4324x^3y^4z^5 + 4xy^6z^5 + 2380x^6z^6 - 1436x^4y^2z^6 + 6x^2y^4z^6 - 152x^5z^7 + 4x^3y^2z^7 + x^4z^8 - 30525$$
$$+313020x^8y^2 + 174078x^6y^4 - 291720x^4y^6 + 74880x^2y^8 + 84400x^9z + 278676x^7y^2z - 420468x^5y^4z + 20576x^3y^6z + 40$$
$$-25880x^8z^2 - 76516x^6y^2z^2 - 148254x^4y^4z^2 + 77840x^2y^6z^2 + 5248y^8z^2 - 29808x^7z^3 - 49388x^5y^2z^3 + 23080x^3y$$
$$+14560xy^6z^3 + 14420x^6z^4 - 7852x^4y^2z^4 + 9954x^2y^4z^4 + 568y^6z^4 + 848x^5z^5 + 92x^3y^2z^5 + 1164xy^4z^5 - 984x^4z^6 + 72$$
$$-2y^4z^6 + 112x^3z^7 - 4xy^2z^7 - 2x^2z^8 + 140625x^8 - 270000x^6y^2 + 172800x^4y^4 - 36864x^2y^6 - 75000x^7z + 36000x$$
$$+46080x^3y^4z - 24576xy^6z - 12500x^6z^2 + 49200x^4y^2z^2 - 19968x^2y^4z^2 - 4096y^6z^2 + 15000x^5z^3 - 10560x^3y^2z$$
$$-3072xy^4z^3 - 2250x^4z^4 - 1872x^2y^2z^4 + 768y^4z^4 - 520x^3z^5 + 672xy^2z^5 + 204x^2z^6 - 48y^2z^6 - 24xz^7 + z^8.$$

We define the *edge surface* of C to be the union of all stationary bisecant lines.

In our example the polynomials of degree 2, 3 and 16 define the edge surface of C.

The quadric cone $x^2 - y^2 - xz = 0$ is a component of the edge surface that does not contribute to the boundary of the convex hull.

The algebraic (Zariski closure of the) boundary of the convex hull of C consists of components of the edge surface and tritangent planes.

Problems

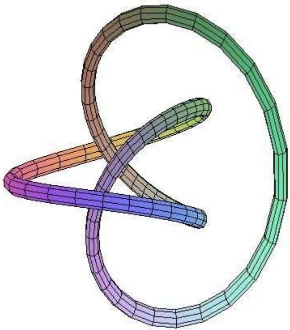
1. How many components of the edge surface and how many tritangent planes are real?

2. How many components of the edge surface and how many tritangent planes contribute to the boundary?

Theorem

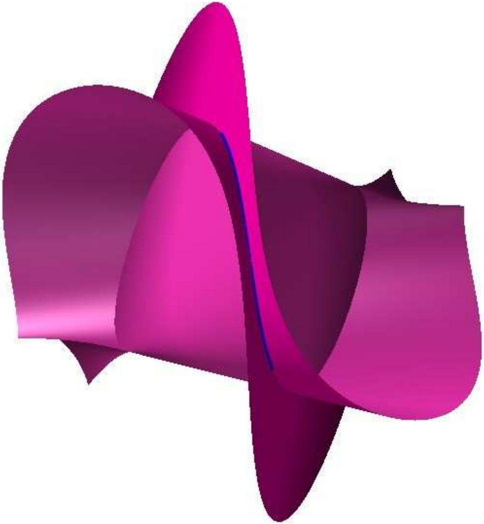
Let C be a general smooth compact curve of degree d and genus g in $\mathbb{R}^3$. The algebraic boundary of its convex hull is the union of the edge surface and tritangent planes. The edge surface is irreducible of degree $2(d-3)(d+g-1)$, and the number of complex tritangent planes is $8\binom{d+g-1}{3} - 8(d+g-4)(d+2g-2) + 8g - 8$.

For a general smooth rational sextic curve, the number of complex tritangent planes is 8.



Morton's curve, $\frac{1}{2-\sin(2\theta)}\left(\cos(3\theta), \sin(3\theta), \cos(2\theta)\right)$ has no real tritangent planes.

Consider irreducible quartic space curves. Of course, they have no tritangent planes.
If a quartic curve is smooth, it is rational or elliptic. If it is singular, then it is rational and has one singular point.



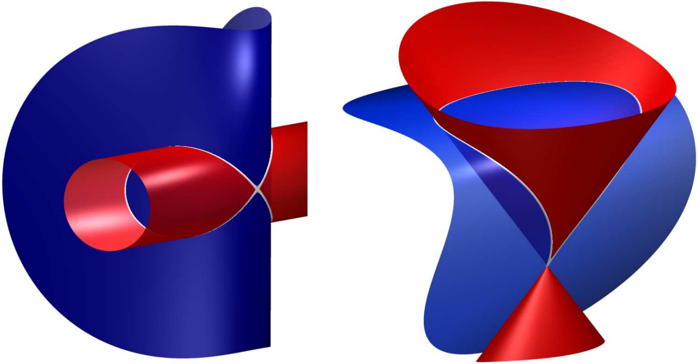
The edge surface of a smooth rational quartic curve is irreducible of degree six.

An elliptic quartic curve is the intersection of a pencil of quadric surfaces. The pencil contains exactly four cones, each with a vertex outside the curve.
The edge surface is the union of these four quadric cones.

A singular (rational) quartic curve has a node or a cusp. It is also the intersection of a pencil of quadric surfaces.

In the nodal case the pencil contains three cones. One has a vertex at the node, the union of the other two form the edge surface.

In the cuspidal case, the pencil contains two cones. Their union form the edge surface.



The edge surface for both a nodal and a cuspidal rational quartic curve is the union of two quadric cones.

Proposition

The variety dual to the edge surface of any space curve is a curve.
In particular, each component of the edge surface is either a cone or the tangent developable of a curve.

A cone is a component of the edge surface if and only if it is a cone of secants with vertex at a cusp, or the general ruling intersects the curve twice outside the vertex.

Problem

Does the edge surface of a smooth space curve have at most one component that is not a cone?

Theorem

The edge surface of a general irreducible space curve of degree d, geometric genus g, with n ordinary nodes and k ordinary cusps, has degree $2(d-3)(d+g-1) - 2n - 2k$.
The cone of bisecants through each cusp has degree $d-2$ and is a component of the surface.

Proofs. Consider the curve of stationary bisecants as a curve B in the symmetric product S^2C. This product has a natural map into the Grassmannian of lines. Classical formulas of Hurwitz and De Jonquiere are used to find the class of B and to compute its degree as a curve in the Grassmannian.
The number of tritangent planes is computed by De Jonquieres formula.

If C is rational, then $S^2C = \mathbb{P}^2$ and the curve B of stationary bitangents is a plane curve.

Trigonometric polynomials $f_1(\theta), f_2(\theta), f_3(\theta)$ define a rational space curve

$$C \;=\; \left\{ \, \big(f_1(\theta), f_2(\theta), f_3(\theta)\big) \in \mathbb{R}^3 \;:\; \theta \in [0, 2\pi] \, \right\} \qquad (0.6)$$

Substituting

$$\cos(\theta) = \frac{x_0^2 - x_1^2}{x_0^2 + x_1^2} \quad \text{and} \quad \sin(\theta) = \frac{2x_0x_1}{x_0^2 + x_1^2} \qquad (0.7)$$

we get rational functions with common denominator $g(x_0, x_1) = (x_0^2 + x_1^2)^d$, for some d.

Multiplying by g we get a parametrized curve in complex projective space

$$\begin{aligned}\bar{C} &= \{(F_0(x) : F_1(x) : F_2(x) : F_3(x)) \\ &= (g : gf_1 : gf_2 : gf_3) \in \mathbb{CP}^3 \ : \ (x_0 : x_1) \in \mathbb{CP}^1 \}.\end{aligned}$$

Given points $p, q \in \bar{C}$, represented by $x_p = (x_{p0} : x_{p1})$ and $x_q = (x_{q0} : x_{q1})$ in $\mathbb{CP}^1$.

$$\begin{pmatrix} F_0(x_p) & F_1(x_p) & F_2(x_p) & F_3(x_p) \\ F_0(x_q) & F_1(x_q) & F_2(x_q) & F_3(x_q) \end{pmatrix}.$$

defines the secant line to $\bar{C}$ through p and q.

The tangent line at p is defined by the partial derivatives

$$\begin{pmatrix} \frac{\partial}{\partial x_{p0}} F_0(x_p) & \frac{\partial}{\partial x_{p0}} F_1(x_p) & \frac{\partial}{\partial x_{p0}} F_2(x_p) & \frac{\partial}{\partial x_{p0}} F_3(x_p) \\ \frac{\partial}{\partial x_{p1}} F_0(x_p) & \frac{\partial}{\partial x_{p1}} F_1(x_p) & \frac{\partial}{\partial x_{p1}} F_2(x_p) & \frac{\partial}{\partial x_{p1}} F_3(x_p) \end{pmatrix}.$$

The secant line between the points p and q is *stationary* if the determinant of the matrix

$$\begin{pmatrix} \frac{\partial}{\partial x_{p0}} F_0(x_p) & \frac{\partial}{\partial x_{p0}} F_1(x_p) & \frac{\partial}{\partial x_{p0}} F_2(x_p) & \frac{\partial}{\partial x_{p0}} F_3(x_p) \\ \frac{\partial}{\partial x_{p1}} F_0(x_p) & \frac{\partial}{\partial x_{p1}} F_1(x_p) & \frac{\partial}{\partial x_{p1}} F_2(x_p) & \frac{\partial}{\partial x_{p1}} F_3(x_p) \\ \frac{\partial}{\partial x_{q0}} F_0(x_q) & \frac{\partial}{\partial x_{q0}} F_1(x_q) & \frac{\partial}{\partial x_{q0}} F_2(x_q) & \frac{\partial}{\partial x_{q0}} F_3(x_q) \\ \frac{\partial}{\partial x_{q1}} F_0(x_q) & \frac{\partial}{\partial x_{q1}} F_1(x_q) & \frac{\partial}{\partial x_{q1}} F_2(x_q) & \frac{\partial}{\partial x_{q1}} F_3(x_q) \end{pmatrix} \tag{0.8}$$

vanishes.

The factor $x_{p0}x_{q1} - x_{p1}x_{q0}$ appears with multiplicity 4 in the determinant. Removing this factor we write the resulting expression as a polynomial $\Phi(a, b, c)$ in the symmetric polynomials

$$a = x_{p0}x_{q0}\,,\;\; b = x_{p1}x_{q1}\,,\;\; c = x_{p0}x_{q1} + x_{p1}x_{q0}. \tag{0.9}$$

$\Phi(a, b, c)$ defines the curve of stationary bisecant lines.

In our first example this polynomial is

$$\Phi \;=\; (a-b)\,c\,(3a^4 - 6a^2b^2 + 2a^2c^2 + 3b^4 + 2b^2c^2 - c^4).$$

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