Theory of Functions of Noncommuting Variables and Its Applications

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1 Overview of the Project

Polynomials, rational functions, and formal power series in (free) noncommuting variables were considered in a variety of settings. While usually viewed as formal algebraic objects, they also appeared often as functions by substituting tuples of matrices or operators for the variables. Our point of view is that a function of noncommuting variables is a function defined on tuples of matrices of all sizes that satisfies certain compatibility conditions as we vary the size of matrices: it respects direct sums and simultaneous similarities, or equivalently, simultaneous intertwinings. This leads naturally to a noncommutative difference-differential calculus. The objective of our research is to develop a comprehensive theory of noncommutative functions and their difference-differential calculus in both algebraic and analytic setting. We expect this theory to have a wide range of applications from noncommutative spectral theory (compare Taylor [8, 9]) and free probability (compare Voiculsecu [10, 11]) to analysis of linear matrix inequalities (LMIs) in optimization and control (compare Helton [1], Helton–McCullough–Vinnikov [2], Helton–McCullough–Putinar–Vinnikov [3]).

2 Preliminary discussion: noncommutative polynomials and noncommutative formal power series

The simplest function of several commuting variables is doubtless a polynomial function that arises by evaluating a polynomial on tuples of (say) complex numbers. Let us consider instead the ring $\mathbb{C}\langle x_1, \ldots, x_d\rangle$ of noncommutative polynomials (the free associative algebra) over $\mathbb{C}$; $x_1, \ldots, x_d$ are noncommuting indeterminates, and $f \in \mathbb{C}\langle x_1, \ldots, x_d\rangle$ is of the form

$$f = \sum_{w \in \mathbb{F}_d} f_w x^w, \tag{1}$$

where $\mathbb{F}_d$ denotes the free semigroup on d generators, $f_w \in \mathbb{C}$, x^w are noncommutative monomials in $x_1, \ldots, x_d$, and the sum is finite. Notice that f can be evaluated in an obvious way on d-tuples of complex matrices of all sizes: for $X = (X_1, \ldots, X_d) \in (\mathbb{C}^{n\times n})^d$,

$$f(X) = \sum_{w \in \mathbb{F}_d} f_w X^w. \tag{2}$$

We can also consider the ring $\mathbb{C}\langle\langle x_1,\ldots,x_d\rangle\rangle$ of noncommutative formal power series (the completion of the free associative algebra) over $\mathbb{C}$; $f\in\mathbb{C}\langle\langle x_1,\ldots,x_d\rangle\rangle$ is of the same form as in (1), except that the sum is in general infinite. There are two ways to evaluate f on d-tuples of complex matrices:

- Assume that $X=(X_1,\ldots,X_d)\in\left(\mathbb{C}^{n\times n}\right)^d$ is a jointly nilpotent d-tuple, meaning that $X^w=0$ for all $w\in\mathbb{F}_d$ with $|w|=k$ for some k, where $|w|$ denotes the length of the word w; equivalently X is jointly similar to a d-tuple of strictly upper-triangular matrices. Then we can define $f(X)$ as in (2), since the sum is actually finite.
- Assume that f has a positive noncommutative multi-radius of convergence, i.e., there exists a d-tuple $\rho=(\rho_1,\ldots,\rho_d)$ of strictly positive numbers such that

$$\limsup_{k\to\infty}\sqrt[k]{\sum_{|w|=k}|f_w|\rho^w}\le 1.$$

  Then we can define $f(X)$ as in (2), where the infinite series converges absolutely and uniformly on any noncommutative polydisc

$$\coprod_{n=1}^{\infty}\left\{X\in\left(\mathbb{C}^{n\times n}\right)^d:\|X_j\|<r_j,\ j=1,\ldots,d\right\}$$

  of mutiradius $r=(r_1,\ldots,r_d)$ with $r_j<\rho_j$, $j=1,\ldots,d$.

We notice that in all these cases the evaluation of f on d-tuples of matrices possesses two key properties.

- f respects direct sums: $f(X\oplus Y)=f(X)\oplus f(Y)$, where

$$X\oplus Y=(X_1,\ldots,X_d)\oplus(Y_1,\ldots,Y_d)=(X_1\oplus Y_1,\ldots,X_d\oplus Y_d)=\left(\begin{bmatrix}X_1&0\\0&Y_1\end{bmatrix},\ldots,\begin{bmatrix}X_d&0\\0&Y_d\end{bmatrix}\right)$$

  (we assume here that X, Y are such that $f(X)$, $f(Y)$ are both defined).
- f respects simultaneous similarities: $f(TXT^{-1})=Tf(X)T^{-1}$, where

$$TXT^{-1}=T(X_1,\ldots,X_d)T^{-1}=(TX_1T^{-1},\ldots,TX_dT^{-1})$$

  (we assume here that X and T are such that $f(X)$ and $f(TXT^{-1})$ are both defined).

3 Overview of some definitions and results

Both for the sake of potential applications and for the sake of developing the theory in its natural generality, it turns out that the proper setting for the theory of noncommutative functions is that of matrices of all sizes over a given vector space (or a given module). In the special case when the vector space is $\mathbb{C}^d$, $n\times n$ matrices over $\mathbb{C}^d$ can be identified with d-tuples of $n\times n$ matrices over $\mathbb{C}$, and we recover noncommutative functions of d variables, key examples of which appeared in the previous section.

Let $\mathcal{V}$ be a vector space over $\mathbb{C}$ (for the algebraic part of the theory, we can consider more generally a module over any commutative ring with unit). We call

$$\mathcal{V}_{\rm nc}=\coprod_{n=1}^{\infty}\mathcal{V}^{n\times n}$$

the noncommutative space over $\mathcal{V}$. A subset $\Omega\subseteq\mathcal{V}_{\rm nc}$ is called a noncommutative set if it is closed under direct sums, i.e., we have

$$X\oplus Y=\begin{bmatrix}X&0\\0&Y\end{bmatrix}\in\Omega_{n+m}$$

for all $X \in \Omega_n$, $Y \in \Omega_m$ and all $n, m \in \mathbb{N}$, where we denote $\Omega_n = \Omega \cap \mathcal{V}^{n\times n}$. Noncommutative sets are the only reasonable domains for noncommutative functions, but additional conditions on the domain are needed for the development of the noncommutative difference-differential calculus. Essentially we need the domain to be closed under formation of upper-triangular block matrices with an arbitrary upper corner block, but this is too much (e.g., this is false for noncommutative polydiscs and balls). The proper notion turns out to be as follows: a noncommutative set $\Omega \subseteq \mathcal{V}_{\text{nc}}$ is called upper admissible if for all $X \in \Omega_n$, $Y \in \Omega_m$ and all $Z \in \mathcal{V}^{n\times m}$, there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that

$$\begin{bmatrix} X & \lambda Z \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}.$$

Our primary examples of upper admissible noncommutative sets are as follows:

- The set $\Omega = \operatorname{Nilp}\mathcal{V}$ of nilpotent matrices over $\mathcal{V}$. Here $X \in \mathcal{V}^{n\times n}$ is called nilpotent if $X^k = 0$ for some k, where we view X as a matrix over the tensor algebra

$$\mathbf{T}(\mathcal{V}) = \bigoplus_{j=0}^{\infty} \mathcal{V}^{\otimes j}$$

  of $\mathcal{V}$; equivalently, there exists $T \in \mathrm{GL}_n(\mathbb{C})$ such that TXT^{-1} is strictly upper triangular.
- Assume that $\mathcal{V}$ is a Banach space and that Ω is open in the sense that $\Omega_n \subseteq \mathcal{V}^{n\times n}$ is open for all n; then Ω is upper admissible.

A special case of the second item — that is crucial for analytic results that are uniform in the size of matrices — is when $\mathcal{V}$ is an operator space This means that there is a sequence of norms $\|\cdot\|_n$ on $\mathcal{V}^{n\times n}$ such that

$$\|X \oplus Y\|_{n+m} = \max\{\|X\|_n, \|Y\|_m\} \quad \text{for all } X \in \mathcal{V}^{n\times n},\ Y \in \mathcal{V}^{m\times m}, \tag{3}$$

and

$$\|TXS\|_n \leq \|T\|\|X\|_n\|S\| \quad \text{for all } X \in \mathcal{V}^{n\times n},\ T, S \in \mathbb{C}^{n\times n}. \tag{4}$$

An important example of an open noncommutative set is then a noncommutative ball

$$\Omega = \coprod_{n=1}^{\infty} \left\{X \in \mathcal{V}^{n\times n} \colon \|X\|_n < \rho\right\}.$$

(For the general theory of operator spaces, see, e.g., Paulsen [6] or Pisier [7].)

Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over $\mathbb{C}$, and let $\Omega \subseteq \mathcal{V}_{\text{nc}}$ be a noncommutative set. A function $f \colon \Omega \to \mathcal{W}_{\text{nc}}$ with $f(\Omega_n) \subseteq \mathcal{W}^{n\times n}$ is called a noncommutative function if:

- f respects direct sums: $f(X \oplus Y) = f(X) \oplus f(Y)$ for all $X \in \Omega_n$, $Y \in \Omega_m$.
- f respects similarities: $f(TXT^{-1}) = Tf(X)T^{-1}$ for all $X \in \Omega_n$ and $T \in \mathrm{GL}_n(\mathbb{C})$ such that $TXT^{-1} \in \Omega_n$.

It turns out that these two conditions are equivalent to a single one: f respects intertwinings, namely if $XS = SY$ then $f(X)S = Sf(Y)$, where $X \in \Omega_n$, $Y \in \Omega_m$, and $S \in \mathbb{C}^{n\times m}$. This condition originates in the pioneering work of Taylor [8].

One can construct noncommutative functions generalizing the formal power series construction discussed in Section 2 to a coordinate free framework. Assume that we are given a sequence $f_k \colon \mathcal{V}^{\otimes k} \to \mathcal{W}$ of linear mappings. Then

$$f(X) = \sum_{k=0}^{\infty} f_k(X^k), \tag{5}$$

where the matrix power X^k is taken in the tensor algebra $\mathbf{T}(\mathcal{V})$ and f_k is extended entrywise to a linear mapping from matrices over $\mathcal{V}^{\otimes k}$ to matrices over $\mathcal{W}$, defines a noncommutative function provided we can make sense of the (generally speaking) infinite sum on the right hand side. This can be done in two ways:

- If X is nilpotent then the sum in (5) is actually finite; hence (5) always defines a noncommutative function on $\mathrm{Nilp}(\mathcal{V})$.
- If $\mathcal{V}$ and $\mathcal{W}$ are operator spaces, and we have a growth estimate

$$\limsup_{k\to\infty} \sqrt[k]{\|f_k\|_{\mathrm{cb}}} \le \frac{1}{\rho}$$

(where $\|\cdot\|_{\mathrm{cb}}$ denotes the completely bounded norm), then the series in (5) converges absolutely and uniformly on any noncommutative ball of radius $r < \rho$; hence in this case (5) defines a noncommutive function on the noncommutative ball of radius ρ.

One of the main results of the noncommutative difference-differential calculus is the infinite series expansion, called the Taylor–Taylor expansion[1], that provides a converse to the above construction. It is given by

$$f(X) = \sum_{k=0}^{\infty} \Delta_R^k f(\underbrace{0,\dots,0}_{k+1})(X^k). \tag{6}$$

Here the multilinear forms $\Delta_R^k f(\underbrace{0,\dots,0}_{k+1})\colon \mathcal{V}^k \longrightarrow \mathcal{W}$ are the values at $(0,\dots,0)$ of the kth order noncommutative difference-differential operators applied to f. They can be calculated directly by evaluating f on block upper trangular matrices:

$$f\left(\begin{bmatrix} 0 & Z_1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 0 & Z_k \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} f(0) & \Delta_R f(0,0)(Z_1) & \cdots & \cdots & \Delta_R^k f(0,\dots,0)(Z_1,\dots,Z_k) \\ 0 & f(0) & \ddots & & \Delta_R^{k-1} f(0,\dots,0)(Z_2,\dots,Z_k) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & f(0) & \Delta_R f(0,0)(Z_k) \\ 0 & \cdots & \cdots & 0 & f(0) \end{bmatrix}.$$

The exact meaning of (6) is one of the two:

- If f is a noncommutative function on $\mathrm{Nilp}(\mathcal{V})$, then the expansion holds for all $X \in \mathrm{Nilp}(\mathcal{V})$.
- If f is a bounded noncommutative function whose domain contains an open noncommutative ball of radius ρ and that is bounded there, then the expansion holds on this ball with the series converging absolutely and uniformly on every noncommutative ball of a strictly smaller radius.

This is merely the simplest of the various convergent Taylor–Taylor series. The expansion can be around any point in $\mathcal{V}_{\mathrm{nc}}$ rather than about 0, providing for the possibility of analytic continuation. In the case $\mathcal{V} = \mathbb{C}^d$ one can obtain stronger results relating to the absolute convergence of the series (2) over the free semigroup, rather than grouping the terms together to obtain a series of homogeneous polynomials as in (6). One can also relax the assumptions of local uniform boundedness over all matrix sizes (with respect to an operator space norm); if a noncommutative function f is locally bounded (or even just locally bounded on slices) in every matrix size, it is still true that its Taylor–Taylor series is locally uniformly convergent in every matrix size (of course the convergence is no longer uniform across matrix sizes). Thus a very weak regularity assumption on a noncommutative function implies already a very strong regularity result.

[1] In honour of Brook Taylor and of Joseph L. Taylor.

4 Progress during the Banff RIT meeting

We are currently working on completing the foundations of the theory of noncommutative functions and their difference-differential calculus, including the preparation of the manuscript [5]. During our week at Banff we made a considerable progress, especially with regard to the detailed proof of the convergence theorem for the Taylor–Taylor series in the non-uniform case, including some facets having to do with the classical theory of analytic functions in several and in infinitely many variables (see, e.g., Hille–Phillips [4] for the later).

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