Random Schrödinger Operators: Universal Localization, Correlations, and Interactions

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1 Overview and Workshop Activities

By many accounts, the quantum mechanics of disordered media had its origin in Anderson's ground-breaking and Nobel Prize winning work in 1958 on what is now called Anderson localization. In mathematics, the quantum mechanical model studied by Anderson, now called the Anderson model, is at the center of the field of random Schrödinger operators. The mathematical analysis of the Anderson, and related, models took off in the second half of the 1970s with the first rigorous proofs of Anderson localization. Even after 30 years of development, the field of random Schrödinger operators remains a focal point of intense mathematical research. Anderson localization also continues to challenge physicists, as evidenced for example in the two papers [52] and [7] in a recent issue of *Physics today*.

There has been significant progress on the mathematics of random Schrödinger operators in recent years. The goal of the workshop on "Random Schrödinger Operators: Universal Localization, Correlations, and Interactions" was to bring leading international experts together to discuss these new developments and to focus the research community on outstanding open problems.

Among the workshops 39 participants were 22 from North America and 17 from overseas. Included were 15 young researchers who have received their PhD degrees in 2002 or later (including some current graduate students).

The workshop activities consisted of the following main parts:

- Nine hour-long talks were delivered by leading experts. Each one of these talks was aimed at surveying recent results in a specific research area. Speakers were asked to stress open problems.
- Seventeen 25-minute talks were used to present recent results, many of them given by younger researchers. Only about half of these talks had been scheduled in advance, with the remaining talks added after the start of the workshop to be able to include very recent developments.
- A two-hour plenary discussion on Wednesday evening focused on possible mathematical approaches to understanding the physically conjectured Anderson transition in three-dimensional disordered media, the central open problem in the theory of random Schrödinger operators.
- In addition to the schedule of plenary lectures, the smaller meeting rooms at BIRS were frequently used by smaller groups to continue existing research collaborations and to plan new joint activities.

In the nine subsections of the following chapter we try to describe some of main research topics which were discussed during the workshop. Much of this relates to the contents of the longer survey talks which were presented at the workshop, but we also include material from some of the shorter talks and of important recent works by researchers who could not be present at the workshop.

Throughout this presentation we stress open problems. In fact, some of the subsections mostly serve the purpose of pointing out problems in the area of random Schrödinger operators where mathematicians have made little or no progress so far. We have also included a number of references to recent works, but not attempted to be comprehensive.

We hope that this report not only describes the current state of the art in the theory of random Schrödinger operators, but will also help focus the efforts of researchers to approach some of the many important open problems which remain.

2 Recent Developments and Open Problems

We define the basic Anderson model on the lattice $\mathbb{Z}^d$ and on the continuum $\mathbb{R}^d$ in order to fix notation. The discrete d-dimensional Anderson model is defined by

$$(h_\omega(\lambda)u)(n) = \sum_{k:|k-n|=1} u(k) + \lambda\omega_n u(n), \quad u \in \ell^2(\mathbb{Z}^d), \tag{1}$$

where $\omega = (\omega_n)_{n\in\mathbb{Z}^d}$ is a d-dimensional array of real-valued independent, identically distributed (i.i.d.) random variables. The parameter $\lambda \geq 0$ is a measure of the disorder. By $|\cdot|$ we denote the 1-norm on $\mathbb{Z}^d$.

The continuum Anderson model is given as

$$H_\omega(\lambda) = -\Delta + \lambda \sum_{n\in\mathbb{Z}^d} \omega_n f(x-n) \quad \text{on } L^2(\mathbb{R}^d). \tag{2}$$

Here the i.i.d. random variables ω_n are as above and $f : \mathbb{R}^d \to \mathbb{R}$ is a fixed single-site potential satisfying suitable decay and regularity assumptions to guarantee summability in (2) and the self-adjointness of H_ω. for both models, we are concerned with the properties of the Hamiltonians for almost every configuration ω.

Various generalizations of the Anderson models as well as a number of other types of random Schrödinger operators have been studied. We will introduce some of them below without intending to be comprehensive.

2.1 Existence of extended states and a mobility edge in d=3

One of the main motivations for the study of the Anderson model is to understand the conjectured Anderson metal-insulator transition. Under reasonable assumptions on the probability measure of the random variables ω_n, the spectrum of $h_\omega(\lambda)$ is $[-2d-\lambda, 2d+\lambda]$ almost surely. The spectral version of this conjecture states that for the model (1) in dimensions $d \geq 3$, and for weak disorder $\lambda \sim 0$, the spectrum near zero energy in the center of the band should be absolutely continuous whereas the spectrum near the band-edges should be pure point with exponentially decaying eigenfunctions. Physically, one expects that there are extended states, corresponding to transport, near the center of the band and localized states near the band-edges. It is known that as the disorder λ increases the spectrum eventually becomes pure point almost surely. Of course, for $\lambda = 0$ the spectrum is absolutely continuous. Numerical calculations, based on the transfer matrix method and performed during the eighties [50] have made these conjectures more precise.

There are several ways to characterize the conjectured Anderson transition. The spectral version is mentioned above. A more physical characterization may be given in terms of solutions to the related time-dependent Schrödinger equation. Dynamical localization refers to the nonspreading of wave packets under the time evolution. Roughly, the random system exhibits dynamical localization in an energy interval Δ if for any state ψ, the evolved state $\psi_\Delta(t) \equiv e^{-itH_\omega}E_{H_\omega}(\Delta)\psi$ remains essentially bounded in time in the sense that the moments of the position operator x remain bounded in time

$$\langle \psi_\Delta(t), |x|^m \psi_\Delta(t)\rangle \leq C_{\psi,\Delta} \tag{3}$$

On the other hand, a transport (or dynamical) transition occurs if there also is an interval Δ' for which some moments are unbounded in time:

$$\langle \psi_{\Delta'}(t), |x|^m \psi_{\Delta'}(t) \rangle \geq \langle t \rangle^\alpha C_{\psi,\Delta'}, \tag{4}$$

for some $\alpha > 0$. Such a transport metal-insulator transition was characterized by Germinet and Klein [34] and proved to occur in the random Landau Hamiltonian near each Landau level (at least at a single energy) by Germinet, Klein and Schenker [36].

Another avenue for characterizing the Anderson transition is the behavior of local eigenvalue statistics (see also Section 2.4 and Section 2.5.) It is known that in the localization regime, the Anderson model have local eigenvalue statistics given by the Poisson process. There is no level repulsion. It is conjectured that if there is a transition, then there is a region of energy for which the statistics follows the Wigner surmise indicating level repulsion. Also, physicists often use the behavior of the inverse participation ratio, defined in terms of the eigenfunctions for the finite volume Hamiltonian by

$$P(\psi; \Lambda) \equiv \frac{\int_\Lambda |\psi|^{2q}}{\int_\Lambda |\psi|^2}, \quad q > 1, \tag{5}$$

to characterize the Anderson transition. In a region of extended states for the infinite-volume Hamiltonian, the inverse participation ratio should scale as $|\Lambda|^{-(q-1)}$, whereas for energies in the region of localization for the infinite-volume Hamiltonian, $P(\psi; \Lambda) \sim |\Lambda|^0$, as $|\Lambda| \to \infty$.

Another way to characterize the transition is to consider the behavior of one of the following quantities: (i) the zero temperature conductivity [50], (ii) the non vanishing of the current-current correlation function at coinciding energies, and (iii) the asymptotic behavior of correlations between Green functions. The current-current correlation function mentioned in (ii) is related to the conductivity through the fluctuation-dissipation theorem. It is also related to the localization length [11], that is, to the moments (3) and (4).

It is known for one-dimensional random Schrödinger operators that whenever the disorder is nonzero $\lambda \neq 0$ the entire spectrum is pure point with probability one. Hence, there is no Anderson transition. Dimension $d = 2$ seems to be the critical dimension [1]. There are no accepted physical arguments for the existence of a transition. For $d > 3$, one expects a transition as in the $d = 3$ case.

In general, nothing is known about extended states for the $d = 3$ Anderson model. Extended states have been proved to exist for the Anderson model on the Bethe lattice [44] (see also more recent results [4] and [31]). There has been progress on other, related models, such as models with decaying randomness, see, for example [13, 14, 43].

Physicists have long considered the Anderson model as a saddle-point approximation to the nonlinear sigma model [27]. Disertori presented an overview of the supersymmetric approach to the nonlinear sigma model and showed how to calculate the density of states and expectations of products of Green's functions. This supersymmetric method has been useful in the study of the density of states for random band matrices [24]. More recently, Disertori, Spencer and Zirnbauer [25] studied a the three-dimensional supersymmetric hyperbolic sigma model and proved the existence of diffusion at low temperature. They argue that this model reflects the Anderson transition expected for random band matrices.

2.2 Non-existence of the mobility edge in d=2

As mentioned above, this is a wide open problem. One possibility is that the spectrum remains pure point (with the eigenfunctions exhibiting power-law rather than exponential decay) but that a transport transition occurs. Such a transport transition has been proved for the two-dimensional Landau Hamiltonian on $L^2(\mathbb{R}^2)$ with an random Anderson-type potential by Germinet, Klein and Schenker [36]. In the eighties, much effort was directed towards proving estimates for the Anderson model on a finite width strip that are uniform in the strip width. There has been some recent interest in this approach via quasi-one-dimensional models and it is discussed in Section 2.8.

2.3 Anderson localization for singular marginal distributions

Bourgain and Kenig [16] made a break-through in proving localization for the continuum Bernoulli-Anderson model. When the random variables are Bernoulli distributed, the usual methods of proving localization break down. The new ideas include a weak Wegner estimate, that is scale dependent, the use of free variables, a new multiscale analysis, and a unique continuation result.

Problem: *The unique continuation result used in [16] does not hold on the lattice and the problem of localization for the discrete Anderson model (1) with Bernoulli distributed ω_n remains open for $d \geq 2$.*

Building on these ideas, Germinet, Hislop and Klein [32, 33] proved localization for the Poisson-Anderson model on $L^2(\mathbb{R}^d)$. The model is given by

$$H_\omega^P = -\Delta + \sum_i f(x - X_i(\omega)) \quad \text{in } L^2(\mathbb{R}^d), \tag{6}$$

where $X_i(\omega)$ are the points of a d-dimensional Poisson process.

Aizenman, Germinet, Klein and Warzel [2] showed that the Bourgain-Kenig result on localization for the continuum Bernoulli-Anderson model yields spectral localization (i.e., just pure point spectrum) for the continuum Anderson model with an arbitrary single-site distribution. (This is based on the observation, originally due to Kolmogorov, that any random variable has a Bernoulli component.) These results have recently been extended by Germinet and Klein [35] to include Anderson localization (i.e., spectral localization with exponentially decayng eigenfunctions) and dynamical localization. These results strengthen our understanding of localization for Anderson-type models.

2.4 Level statistics of eigenvalues in the Anderson model

Level statistics of eigenvalues was noted long ago by physicists as a relevant method to characterize the extended states regime [6, 26, 38]. Since then, eigenvalue statistics has become a standard method for numerically analyzing the Anderson and related transitions and to distinguish between the extended and localized regimes.

More precisely, let E_n^Λ be the eigenvalues of a restriction of a random Hamiltonian to a finite box Λ with self-adjoint boundary conditions. Because the average eigenvalue spacing scales as the volume of Λ, we consider the rescaled eigenvalues $\xi_n^\Lambda = (E_n^\Lambda - E)|\Lambda|$ near a given energy level E. If E is in a region of the spectrum that exhibits localization, then for large Λ the point process given by ξ_n^Λ should approach a Poisson point process. On the other hand, extended states near E should be characterized by level repulsion between the ξ_n^Λ, which is expected to be described by level statistics as found, for example, in Gaussian unitary ensembles of random matrices.

Rigorous results on level statistics for random Schrödinger operators are currently only known in the localized energy regime. Molchanov [54] proved Poisson statistics for a one-dimensional continuum random Schrödinger operator with potential generated by Brownian motion on a compact Riemannian manifold. Minami [53] proved Poisson statistics for the discrete Anderson model in any dimension.

At the workshop, Combes reported on recent joint work with Germinet and Klein [20] that extends Minami's results on the Poisson nature of the local eigenvalue statistics to continuum Anderson models for energy intervals near the bottom of the spectrum by using a new approach (recent progress [21] has resulted in the removal of some of the technical assumptions of [20].) One of the important tools in the proof of Poisson statistics is the *Minami estimate*. For lattice Anderson models, Minami proved that there is a universal constant C_M such that, for arbitrary intervals I and arbitrary finite boxes Λ,

$$\mathbb{P}(I \text{ contains at least two } E_n^\Lambda) \leq C_M |I|^2 |\Lambda|^2. \tag{7}$$

At the moment, the result of [20] in the continuum requires that I be close to the bottom of the deterministic spectrum, and some additional technical assumptions. This is sufficient to conclude Poisson statistics in this energy regime. The Minami estimate is a second-order correlation function. It should be compared with the Wegner estimate

$$\mathbb{P}(I \text{ contains one } E_n^\Lambda) \leq C_W |I||\Lambda|, \tag{8}$$

which is a first-order correlation function. Thus the Minami estimate says that the probability for finding two eigenvalues in a small interval is roughly the square of the probability for finding one eigenvalue, an indication of independence of energy levels. The following problem remains open:

Problem: *Determine if a Minami estimate (7) holds for the continuum Anderson model at all energies, where the constant C_M can be chosen uniformly for all $I \subset (-\infty, E_{max}]$ (and may grow with E_{max}). Also find bounds on higher eigenvalue correlations of the form*

$$\mathbb{P}(I \text{ contains } n \text{ of the eigenalues } E_n^\Lambda) \leq C(n, E_{max})|I|^n|\Lambda|^n. \tag{9}$$

At the workshop we also had a report by Stoiciu on work on level statistics for the eigenvalues of random CMV-matrices. His work [58] established Poisson statistics for CMV-matrices with Verblunsky coefficients given by i.i.d. random variables with rotationally invariant distributions. In joint work with Killip [42], he also studied the case of decaying random Verblunsky coefficients, where with increasing decay rate a transition from Poisson statistics to clock-type statistics can be observed. They also observe a transitional type of statistics at a critical decay rate.

Due to the general lack of rigorous understanding of the extended states regime in the multi-dimensional Anderson model there are, consequently, no results establishing level repulsion in any energy and disorder regimes for the Anderson model. There do not even seem to be conditional results, saying that level repulsion would imply one or more of the other possible characterizations of extended states described in Section 2.1 above, or, vice versa, level repulsion would follow from them.

2.5 Localization and delocalization for random band matrices

At the workshop Erdös reported on results on the universality of the gap distribution and k-point correlations for hermitian Wigner random matrices. Such results were recently proven by several authors to hold in great generality. The most general result is given in [28], where the methods used in [29] and [59] to prove similar results under more restrictive assumptions were combined. Related results are also presented in [39].

A hermitian Wigner random matrix is an $N \times N$ hermitian matrix $H = (h_{\ell k})$ with i.i.d. entries $h_{\ell k}$. More precisely, $h_{\ell k} = N^{-1/2} z_{\ell k}$ with $z_{\ell k} = x_{\ell k} + i y_{\ell k}$. For $\ell < k$ the $x_{\ell k}$ and $y_{\ell k}$ are i.i.d. random variables with distribution ν of expectation zero and variance $1/2$. On the diagonal one has $y_{\ell\ell} = 0$ and $x_{\ell\ell}$ i.i.d. with distribution $\tilde{\nu}$ of expectation zero and variance 1.

A classical special case is the Gaussian unitary ensemble (GUE) where $x_{\ell k}$, $y_{\ell k}$ and $x_{\ell\ell}$ are Gaussian random variables with mean zero and variance $1/2$ and 1, respectively. In this case the eigenvalue density is explicitly known since Wigner to be given by the semicircle law $\sqrt{(4-x^2)_+}/2\pi$ and the k-point correlation functions are determinants of the Dyson sine kernel

$$K(x, y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}.$$

The long standing conjecture of *bulk universality* means that for general Wigner random matrices, on the bulk $(-2, 2)$ of the spectrum and as $N \to \infty$, the eigenvalue density and k-point correlation functions should converge, in suitable sense, to those for the GUE case. In [28] such results and related results for the gap distribution are shown under merely the additional assumption of subexponential decay of the laws ν and $\tilde{\nu}$, i.e. that $\nu(\mathbb{R} \setminus (-t^C, t^C)) \leq e^{-t}$ for some constant C and similar for $\tilde{\nu}$.

Wigner ensembles on one hand and Anderson models on the other represent opposite extremes of models of random matrices. While in a Wigner ensemble all entries are random variables, in the Anderson model only entries on or, in related models, near the diagonal are random, with most off-diagonal entries vanishing. Nevertheless, it is expected that the level statistics observed for Wigner ensembles might also be relevant for Anderson models. More precisely, the fact that the 2-point correlation function for Wigner ensembles is governed by the Dyson sine kernel establishes a specific type of level repulsion. It is expected that the same law of level repulsion holds in the conjectured extended states regime of the three-dimensional Anderson model. Just like all other possible incarnations of extended states in the Anderson model this is a wide open problem. Somewhat more accessible might be the

Problem: *Consider models of hermitian $N \times N$ random band matrices $H = (h_{\ell k})$, where $h_{\ell k} = 0$ for $|\ell - k| \geq CN^\rho$, $1 > \rho > 1/2$, and the non-vanishing entries are suitably normalized independent random variables. Show results similar to those in [28], in particular, establish level repulsion of the eigenvalues for* *large N. For $\rho < 1/2$, on the other hand, one expects that random band matrices have localized eigenvectors and that their eigenvalue statistics are close to a Poisson process.*

Some results towards the last part of this problem were established in [56]. In particular, it is shown there that Gaussian band ensembles have localized eigenvectors if $\rho < 1/8$. Schenker [56] also provides further discussion and some physics references for the expected results on localization and level statistics of random band matrices.

2.6 N-body systems and the effects of electron-electron interactions

All the results discussed so far concern one-particle systems, or equivalently non-interacting particle systems. Much less well understood are localization properties of systems of interacting particles subject to an external random potential. Most interesting would be an understanding of N interacting fermions in a region of volume $|\Lambda|$ with $N \to \infty$ and $N/|\Lambda| \to \rho > 0$. If the interactions are weak and the particle density ρ small, then it is expected that localization phenomena as in one-particle systems will persist. Mathematically, this is a major open problem which currently does not seem to be within reach.

Progress has been made recently for the case of a fixed number N of interacting particles and $|\Lambda| \to \infty$. At the workshop, Chulaevsky reported on his work with Suhov [18, 19] where a Wegner estimate and the first multi-particle localization results on Anderson localization were first established for a 2-interacting particle Anderson-type model. These results were later extended to N particles, also by Aizenman and Warzel [5], who also proved dynamical localization. These papers consider an N-particle Anderson-type model, acting on $\phi \in \ell^2(\mathbb{Z}^{Nd})$ via the Hamiltonian

$$(h_\omega^{(N)}\phi)(x) = \sum_{y:|y-x|=1} \phi(y) + (U(x) + g\sum_{j=1}^{N} \omega_{x_j})\phi(x), \tag{10}$$

where $x = (x_1, \ldots, x_N) \in \mathbb{Z}^{Nd}$, $y \in \mathbb{Z}^{Nd}$, and $|\cdot|$ now denotes the 1-norm on $\mathbb{Z}^{Nd}$. While both papers can handle more general situations, we only mention two particle interactions

$$U(x) = \sum_{1 \le j < k \le N} \Phi(x_j - x_k)$$

with a function Φ of finite support, which are covered in both papers.

Both papers establish localization in the large disorder regime, i.e. for sufficiently large g. For this case Chulaevsky and Suhov show Anderson localization: $h_\omega^{(N)}$ almost surely has pure point spectrum with exponentially decaying eigenfunctions. Aizenman and Warzel prove an N-particle version of dynamical localization, see (3) above, which implies Anderson localization. They also can show localization for the interacting particle system if the underlying one-particle system is localized and the interaction strength is sufficiently small. For example, this applies in the case of interacting one-dimensional particles ($d = 1$).

The method underlying [19] is an extension of multiscale analysis to N-particle systems, also using corresponding extensions of Wegner estimates. In [5], on the other hand, the central idea is an extension of the fractional moments method, originally developed for one-particle systems in [3]. One consequence of the use of different methods is that [19] can work with weaker regularity assumptions on the distribution of the random couplings ω_n (Hölder continuity as opposed to absolute continuity with bounded density in [5]). The approach by Chulaevsky and Suhov has also recently been extended in [17] to cover continuum Anderson-type models of interacting particles, based on the one-particle Hamiltonian (2).

In addition to the hard problem of understanding localization properties of infinite systems of positive particle density, we mention two other open problems which should be more easily accessible:

Problem: *Is (10) localized at all energies for $d = 1$, independent of the disorder strength and the size of the interaction potential? For small interaction potential this is shown in [5]. But for a finite number of one-dimensional particles, one might expect to only need the assumption that Φ has finite support (or decays sufficiently fast). In trying to prove this, it might help to assume that the particle interaction is repulsive (as it is for electrons), an assumption which is not used in [19] and [5].*

Problem: *Do the results on level statistics from Section 2.4 extend to interacting particle systems? In particular, are the eigenvalue levels Poisson distributed in the case of large disorder? Klopp and Zenk [49] have recently shown that the restriction of (10) to a finite box $\Lambda \subset \mathbb{Z}^d$ satisfies a Wegner estimate (8) with right hand side $C|I||\Lambda|^N$. Proving Poisson statistics for interacting systems would also require to have a Minami estimate (7) with right hand side $C|I|^2|\Lambda|^{2N}$.*

We also mention that Shepelyansky [57] showed on a physical level of rigor that the localization length for two-particle random systems is strongly enhanced in some energy intervals. There is some experimental evidence for this effect, see [41], that has yet to be investigated mathematically.

2.7 Models of non-monotonic random operators

The discrete Anderson model h_ω as well as the continuum Anderson model H_ω with sign-definite single-site potential depend monotonically on the random parameters ω_n: If $\omega = (\omega_n)_{n\in\mathbb{Z}^d}$ and $\tilde{\omega} = (\tilde{\omega}_n)_{n\in\mathbb{Z}^d}$ are two random configurations such that $\omega_n \le \tilde{\omega}_n$ for all n, then the quadratic form q_ω associated with h_ω or H_ω satisfies

$$q_\omega(\varphi, \varphi) \le q_{\tilde{\omega}}(\varphi, \varphi)$$

for all φ in its domain (where we assume positive single site potential for the case of (2)). This leads to many important technical advantages in the theory of these models, for example in proofs of Wegner estimates and spectral averaging properties (see, for example, [22].)

The continuum Anderson model becomes non-monotonic if the single site potential f is sign-indefinite. Another example of a non-monotonic random operator is the Poisson model (6) discussed in Section 2.3. Here the randomness appears in the form of *structural disorder* in the positions of the single-site potentials, a property which it shares with the *random displacement model*. The latter has the form

$$H_\omega^D = -\Delta + \sum_{n\in\mathbb{Z}^d} f(x - n - d_n), \qquad (11)$$

where the d_n are i.i.d. vector-valued random variables describing the displacement of the random scatterers from the sites of the perfect lattice $\mathbb{Z}^d$.

Non-monotonic random Schrödinger operators pose many technical challenges. Understanding their properties often requires better understanding of the underlying physics. For example, due to the monotonicity in the potential energy, proving Wegner estimates or spectral averaging properties for sign-definite Anderson models generally requires very little use of the properties of the Laplacian, i.e. the kinetic energy operator. In the absence of an obvious monotonicity property of the potential energy, much more use of the physical interaction between kinetic and potential energy has to be made.

Recent progress on non-monotonic random operators reported at the workshop includes the following:

- Klopp and Nakamura [47, 48] have carried out a detailed study of the low energy properties of the sign-indefinite Anderson model with reflection symmetric single-site potential (i.e. f is symmetric in each coordinate while the other coordinates are kept fixed). They have results on the characterization of the spectral minimum and on the asymptotic behavior of the integrated density of states at the spectral minimum. In many cases they obtain Lifshits-tail behavior of the IDS, where they can combine this with earlier work on Wegner estimates for the sign-indefinite Anderson model [40] to conclude localization at low energies.

- For the case of the random displacement model the spectral minimum and asymptotics of the IDS were studied by Baker, Loss and Stolz [9, 10], also under the assumption of reflection symmetric single site potential. For the random displacement model, as well as for the indefinite Anderson model, the low energy asymptotics of the IDS show a surprisingly rich variety of phenomena. One may obtain Lifshits tails, van Hove asymptotics, or, for the one-dimensional displacement model, even a logarithmic singularity.

- Elgart, Tautenhahn and Veselic [30] obtain exponential decay of suitable fractional moments of Green's function for a discrete version of the indefinite Anderson model. This does not require symmetry of the single site potentials, but is currently restricted to $d = 1$ and requires overlapping single site potentials.

Some time back Klopp [46] established localization for a semiclassical version of the random displacement model near the bottom of the spectrum, requiring sufficiently small semi-classical parameter. However, without use of a small semi-classical parameter this remains an open problem.

Problem: *Consider the random displacement model H_ω^D for dimension $d \geq 2$. Assume that the single-site potential f is reflection symmetric with supp $f \subset [-r, r]^d$, $r < 1/2$, and that the random displacements are (for example) uniformly distributed in $[-d_{max}, d_{max}]^d$. Finally, assume $r + d_{max} \leq 1/2$, i.e. non-overlapping sites. Show that H_ω^D is localized for energies near the bottom of its almost sure spectrum.*

Recent progress, including progress made at the workshop, may put the solution of this problem within reach.

2.8 Quasi-one-dimensional random operators

Quasi-one-dimensional random operators and, in particular, Anderson-type models on strips of finite width provide situations in which one can attempt to study the transition between (rather well understood) one-dimensional and (not so well understood) multi-dimensional localization phenomena. Localization for the Anderson model on a discrete strip $\{1, \ldots, N\} \times \mathbb{Z}^{d-1}$ was proven two decades ago for arbitrary width N, e.g. [37] and [45]. At the workshop similar results for one-dimensional continuum random Schrödinger operators with $N \times N$-matrix-valued random potentials were discussed by Boumaza [12]. This can be considered as an Anderson model on a semi-discretized strip consisting of N continuous channels. It is interesting to compare this with results by Sadel and Schulz-Baldes [55], also presented at the workshop, which show that the spectral type for related classes of matrix-valued random Dirac systems depends strongly on the number of channels N. For even N localization is expected, while odd N, under suitable assumptions including time-reversal invariance, leads to purely absolutely continuous spectrum. For all these models it is not understood how the localization characteristics change as the width N becomes large. In particular, to have a chance to draw any conclusion about localization properties in $d = 2$ it would be necessary to correctly determine how the localization length depends on N. Somewhat surprisingly, understanding localization on continuum strips is still an open challenge:

Problem: *Let*

$$H_\omega = -\Delta + \sum_{n \in \mathbb{Z}} \omega_n f(x - n)$$

on the tube $S = [0, 1]^{d-1} \times \mathbb{R}$ with any standard boundary condition on the boundary of the tube. Here f is a single-site potential supported on $[0, 1]^d$. Under suitable assumptions of f and the i.i.d. random parameters ω_n, prove that H_ω is localized at all energies.

This problem highlights the difference between "physical one-dimensionality" and "mathematical one-dimensionality". The system is physically one-dimensional which should yield localization at all energies and under rather weak assumptions. However, mathematically it is governed by a PDE which for non-trivial f can not be reduced to a finite system of coupled ODEs, eliminating all the well-established "mathematically one-dimensional" techniques from the theory of dynamical systems.

2.9 Pseudorandom potentials

A one-dimensional ergodic Schrödinger operator is defined by

$$(h_\omega u)(n) = u(n + 1) + u(n - 1) + \lambda f(T^n \omega) u(n), \quad u \in \ell^2(\mathbb{Z}), \tag{12}$$

where $\omega \in \Omega$, a probability space, T is an ergodic, measure-preserving transformation on Ω, and λ a real parameter measuring disorder strength. The most studied examples of ergodic operators are quasi-periodic operators and the Anderson model. Other classes of dynamical systems (Ω, T) together with suitable choices of f and λ are expected to lead to operators h_ω with spectral properties similar to the Anderson model. Pragmatically, and somewhat vaguely, the corresponding potentials $\lambda f(T^n \omega)$ have been called "pseudorandom potentials". Some results and many open problems relating to such potentials were discussed at the workshop by Damanik, e.g. [8], [23].

Two concrete classes of ergodic potentials which are expected to be pseudorandom are potentials generated by the skew shift or by the doubling map. The skew shift on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is given by

$$T(\omega_1, \omega_2) = (\omega_1 + \omega_2, \omega_2 + \alpha)$$

for an irrational α. An interesting case is given if $f = g \circ P_1$, where P_1 is the projection onto the first coordinate in $\mathbb{T}^2$ and g a function on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. In this case the potential in (12) becomes

$$\lambda f(T^n \omega) = \lambda g(\frac{n(n-1)}{2}\alpha + n\omega_2 + \omega_1). \tag{13}$$

Existing results for the corresponding operators h_ω were discussed at the workshop by Goldstein, see e.g. [15]. Many problems for potentials defined through skew shifts remain open. For example, so far only partial answers are known to the following

Problem: *Fix any $\lambda > 0$ and irrational α and assume that g is sufficiently smooth. Show that the spectrum of h_ω with potential given by (13) is an interval or, at worst, a union of finitely many intervals. Also show positivity of the Lyapunov exponent for all energies.*

A related result was demonstrated at the workshop by Krüger [51], who considered operators of the form

$$(hu)(n) = u(n+1) + u(n-1) + g(n^\rho)u(n),$$

where g is continuous on $\mathbb{T}$. If $\rho > 0$ is not an integer, then he shows that the spectrum of h is an interval. The potential (13) is most closely related to the case $\rho = 2$, where the corresponding question remains open.

Another interesting example are potentials $\lambda f(T^n\omega)$ defined by the doubling map $T\omega = 2\omega$ mod 1 on $\mathbb{T}$. As for the problem for the skew shift stated above, the spectrum in this case is conjectured to be an interval or finite union of intervals and the Lyapunov exponent should be positive for all energies and any disorder λ. Again, only partial results are known. In particular, it is not known if the spectrum is an interval for large λ and if the Lyapunov exponent is positive for small λ.

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