Strong dependence, weight, etc

Alex Usvyatsov

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1 Preliminaries

- Preliminaries.

Basic notations

- *We assume that the theory T is dependent and $T = T^{eq}$.*
- We write $a \equiv_A b$ for $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$.
- We say that a and b are of Lascar distance 1 over a set A if there exists an A-indiscernible sequence containing both. This is not an equivalence relation, but its transitive closure $E^L_A(x, y)$ is. We say that a and b have the same Lascar type if they are E^L_A-equivalent.
- We write $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ or $a \equiv_{\mathrm{Lstp},A} b$.

Basic notations - II

- Let I be an indiscernible sequence over a set A. Then $a \models \mathrm{Av}(I, A \cup I)$ if and only if $I^\frown\{a\}$ is indiscernible over A.
- For I an indiscernible sequence over A, we often denote $\mathrm{Av}(I, A \cup I)$ by $\mathrm{Av}(I)$.

  So this is just the type of the "next element" of I over A.

Basic definitions - forking

- A formula $\varphi(x, a)$ *divides* over a set A if there exists an A-indiscernible sequence $I = \langle a_i : i < \omega \rangle$ containing a such that the set

  $$\{\varphi(x, a_i) \colon i < \omega\}$$

  is inconsistent.
- $\varphi(x, a)$ *forks* over A if it implies a finite disjunction of formulas that divide over A.
- Equivalently, $\varphi(x, a)$ *forks* over A if every global type p which contains $\varphi(x, a)$ divides over A.
- A type p divides/forks over a set A if it contains a dividing/forking formula.

Basic definitions (splitting)

- A type $p \in \mathrm{S}(B)$ *does not split* over a set A if whenever $b, c \in B$ have the same type over A, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.
- A type $p \in \mathrm{S}(B)$ *does not split strongly* over a set A if whenever $b, c \in B$ are of Lascar distance 1 over A, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.
- A type $p \in \mathrm{S}(B)$ *does not Lascar-split* over a set A if whenever $b, c \in B$ have the same Lascar type over A, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.
- Note that a global type doesn't split over a set A if it is invariant under the action of the automorphism group of $\mathfrak{C}$ over A.

Basic definitions (Morley sequences)

- Let O a linear order, A a set. We call a sequence $I = \langle a_i : i \in O \rangle$ a *Morley sequence over* A if it is an indiscernible sequence over A and $\mathrm{tp}(a_i/Aa_{<i})$ does not fork over A for all $i \in O$.
- If a sequence I is indiscernible over B and Morley over $A \subseteq B$, we sometimes say that I is *based on* A.
- Let $p \in \mathrm{S}(B)$ be a type. We call a sequence I a *Morley sequence in* p if it is a Morley sequence over B of realizations of p.
- Let $I = \langle b_i : i < \omega \rangle$ be an indiscernible sequence in $p \in \mathrm{S}(A)$. The following are equivalent:
  - ♢ I is a Morley sequence in p.
  - ♢ $\mathrm{Av}(I) = \mathrm{Av}(I, I \cup A)$ is a nonforking extension of p.
  - ♢ There exists a global extension of $\mathrm{Av}(I)$ which does not fork over A.

Strong splitting and dividing

- (Shelah) Strong splitting implies dividing, hence forking
- Hence Lascar-splitting implies forking (follows since for global types strong splitting coincides with Lascar-splitting).

Important consequences:

- There are boundedly many global types which do not fork over a given set A.
- Let $I = \langle a_i : i < \lambda \rangle$ be such that
  - $\mathrm{tp}(a_i/Aa_{<i})$ does not fork over A
  - $\mathrm{Lstp}(a_i/Aa_{<i}) = \mathrm{Lstp}(a_j/Aa_{<i})$ for every $j \geq i$.

  Then I is a Morley sequence over A (that is, it is indiscernible over A).

Forking - equivalences

The following are equivalent for a global type p and a set A:

- p forks over A
- p divides over A
- p splits strongly over A
- p Lascar splits over A
- p is not Lascar-invariant over A

Morley sequences in dependent theories:

- *Observation.* Let $I = \langle a_i : i < \lambda \rangle$ be such that
  - $\mathrm{tp}(a_i/Aa_{<i})$ does not fork over A
  - $\mathrm{Lstp}(a_i/Aa_{<i}) = \mathrm{Lstp}(a_j/Aa_{<i})$ for every $j \geq i$.

  Then I is a Morley sequence over A (that is, it is indiscernible over A).
- *Proposition.* Let I be a Morley sequence over a set A. Then there exists a unique global types extending $\mathrm{Av}(I)$ which does not fork over A. In other words, $\mathrm{Av}(I)$ is stationary over A.
- We call this global type the *eventual type* of I, $\mathrm{Ev}(I)$.
- One can provide an explicit construction of $\mathrm{Ev}(I)$ (similar to Poizat's eventual type of "special sequences").

Morley sequences in dependent theories II:

- *"Weak Kim's Lemma" (Onshuus, U.).* Let A be an extension base (e.g. a model) and $\varphi(x, a)$ a formula which divides over A. Then there exists a Morley sequence I in $\mathrm{tp}(a/A)$ which witnesses dividing, that is, $\varphi(x, I)$ is inconsistent.
- *Theorem (Chernikov, Kaplan).* Let M be a model, and $\varphi(x, a)$ a formula which forks over M. Then $\varphi(x, a)$ *divides* over M.

Weight

We recall the notion of *weight* in stable theories.

- Let $p(x)$ be any type over some set A. We say that $a, \langle b_i \rangle_{i=1}^{\alpha}$ witnesses (*pre-weight of* p is at least α) if $a \models p(x)$, $\langle b_i \rangle_{i=1}^{\alpha}$ is A-independent and $a \not\perp_A b_i$ for all i.
- The *pre-weight* of p is the supremum over all α such that such a witness exists.
- The *weight* of a type p is defined to be the supremum over the pre-weights of all nonforking extensions of p.

Rudimentarily finite weight

- A type p has *rudimentarily finite pre-weight* if there is no $\langle b_i \rangle_{i=1}^{\omega}$ witnessing that pre-weight of p is at least ω.
- Note that a priori this does not mean that the pre-weight of p is *finite* - there can be witnesses $\langle b_i \rangle_{i=1}^{n}$ for arbitrary large $n < \omega$.
- *Rudimentarily finite weight* is defined similarly.
- (Hyttinnen, Pillay) In a stable theory, a type p which has rudimentarily finite weight, has finite weight. In fact, such p is domination equivalent to a free product of finitely many types of weight 1.

2 Stability in broader contexts

- Stability in unstable contexts.

Stable types

Recall: a (partial) type p is called *stable* if every extension of it is definable.
The following are equivalent for a theory T:

- p is stable.
- For every $B \supseteq A$, p has at most $|B|^{\aleph_0}$ extensions in $\mathrm{S}(B)$.
- There is no formula $\varphi(\bar{x}, \bar{y})$ (with parameters from $\mathfrak{C}$) exemplifying the order property with respect to indiscernible sequences $I = \langle \bar{a}_i : i < \omega \rangle$ and $J = \langle \bar{b}_i : i < \omega \rangle$ with $\cup J \subseteq p^{\mathfrak{C}}$. We call this "$p$ does not admit the order property".

A "stable set" is often referred to as "stable and stably embedded".

Stable types in dependent theories

(Onshuus, Peterzil) Let $p \in \mathrm{S}(A)$. The Following Are Equivalent for a type in a dependent theory:

1. p is stable.
2. For every $B \supseteq A$, p has at most $|B|^{\aleph_0}$ extensions in $\mathrm{S}(B)$.
3. Every indiscernible sequence in p is an indiscernible set.
4. There is no "order property" on p (with or without external parameters)
5. On the set of realizations of p there is no definable (with or without external parameters) partial order with infinite chains.

Example

- Let us consider the theory of $\mathbb{Q}$ with a predicate P_n for every interval $[n, n+1)$ ($n \in \mathbb{Z}$) and the natural order $<_n$ on P_n. It is easy to see that the "generic" type "at infinity" (that is, the type of an element not in any of the P_n's) is stable.

Stable domination

(Hrushovski, Haskell and Macpherson)

- A type $p \in \mathrm{S}(A)$ is called *stably dominated* if there exists a collection of stable sets $\bar{D} = \langle D_i : i < \alpha \rangle$ and definable functions $f_i : p^{\mathfrak{C}} \to D_i$ such that for every set $B \supseteq A$ and $\bar{a} \models p$, if $f_i(\bar{a}) \mathop{\smile\!\!\!\!\!\!|}^{st}_A B$ for all i (which in this context just means that $\mathrm{tp}(f_i(a)/B)$ is definable over A), then (denoting $\bar{f} = \langle f_i : i < \alpha \rangle$) $\mathrm{tp}(f(\bar{a})/B) \vdash \mathrm{tp}(\bar{a}/B)$.

Example

- Let us consider the theory of a two-sorted structure (X, Y): on X there is an equivalence relation $E(x_1, x_2)$ with infinitely many infinite classes and each class densely linearly ordered, while Y is just an infinite set such that there is a definable function f from X onto Y with $f(a_1) = f(a_2) \iff E(a_1, a_2)$.

  In other words, Y is the sort of imaginary elements corresponding to the classes of E.

  Let M a model and p the "generic" type in X over M, that is, a type of an element in a new equivalence class. It is easy to see that p is stably dominated, but clearly not stable.

- Note that the first example shows a stable type which is not stably dominated (there are no stable sets).

Generic stability

Let T be dependent.

- We call a type $p \in \mathrm{S}(A)$ *generically stable* if there exists a Morley sequence $\langle b_i : i < \omega \rangle$ in p (over A) which is an indiscernible set.

- A generically stable type over $A = \mathrm{acl}(A)$ is definable and stationary. In particular, every two Morley sequences in it have the same type.

- A type p is generically stable if and only if there is a Morley sequence I in p such that $\mathrm{Av}(I, \mathfrak{C})$ does not fork over the domain of p if and only if $\mathrm{Av}(I, \mathfrak{C}) = \mathrm{Ev}(I)$ for some/every Morley sequence in p.

More characterizations

The following are equivalent for an extensible type $p \in \mathrm{S}(A)$:

- There exists a Morley sequence $\langle b_i : i < \omega \rangle$ in p (over A) which is an indiscernible set.

- Every Morley sequence $\langle b_i : i < \omega \rangle$ in p (over A) is an indiscernible set.

- Nonforking is symmetric on the set of realizations of p.

- For every b such that $\mathrm{tp}(b/A)$ is extensible, we have $a \mathop{\smile\!\!\!\!\!\!|}_A b \iff b \mathop{\smile\!\!\!\!\!\!|}_A a$.

- Nonforking is a stable independence relation on the set of realizations of p.

- p has a global nonforking extension which is both definable over and finitely satisfiable in a countable Morley sequence I.

Examples

- Every stable type is generically stable.
- Every stably dominated type is generically stable.

More interesting Examples

Generically stable types which are not stable or stably dominated:

- Similar to Example I: $(Q, P_0, <_0, +)$, p the "infinity" type. Then it is generically stable, but there is a definable order on it, so it is unstable.
- *Caution:* unlike stable types, we don't know anything about forking extensions of generically stable types. This is why it is not generally the case that there is a bound of the length of a forking chain of generically stable types. Similarly, generically stable types are not closed under concatenation.

3 Strong dependence

- Strong dependence.

Definitions

- A theory T is *not* strongly dependent if there exists an array $\langle \bar{a}_i^\alpha : i < \omega, \alpha < \omega \rangle$ and formulas $\langle \varphi_\alpha(\bar{x}, \bar{y}_\alpha) : \alpha < \omega \rangle$ (note that $\bar{x}$ does not depend on α) such that for every $\eta \in {}^\omega\omega$ the set

$$\left\{ [\varphi_\alpha(\bar{x}, \bar{a}_i^\alpha)]^{(\eta(\alpha)=i)} : \alpha < \omega, i < \omega \right\}$$

is consistent.

- One can add in addition that $I^\alpha = \langle \bar{a}_i^\alpha : i < \omega \rangle$ is indiscernible over $\cup\{I^\beta : \beta \neq \alpha\}$ for every $\alpha < \omega$. Then there is no need to demand "for all η", it is enough to say, for example:

$$\{[\varphi_\alpha(\bar{x}, \bar{a}_0^\alpha) \wedge \neg\varphi_\alpha(\bar{x}, \bar{a}_1^\alpha)] : \alpha < \omega\}$$

is consistent.

Connections to dependence

Exercises:

- A theory T is independent if and only of there exists an array $\langle \bar{a}_i^\alpha : i < \omega, \alpha < \omega \rangle$ and a formula $\varphi(\bar{x}, \bar{y})$ (so it does not depend on α) such that for every $\eta \in {}^\omega\omega$ the set

$$\left\{ [\varphi(\bar{x}, \bar{a}_i^\alpha)]^{(\eta(\alpha)=i)} : \alpha < \omega, i < \omega \right\}$$

is consistent.

- A theory T is independent if and only of there exists an array $\langle \bar{a}_i^\alpha : i < \omega, \alpha < |T|^+ \rangle$ and formulas $\langle \varphi_\alpha(\bar{x}, \bar{y}_\alpha) : \alpha < |T|^+ \rangle$ such that for every $\eta \in {}^\omega\omega$ the set

$$\left\{ [\varphi_\alpha(\bar{x}, \bar{a}_i^\alpha)]^{(\eta(\alpha)=i)} : \alpha < |T|^+, i < \omega \right\}$$

is consistent.

Cutting indiscernibles - dependence

Theorem (Shelah) The following are equivalent for a theory T:

- T is dependent.
- For every set A, an infinite A-indiscernible sequence I, a finite tuple $\bar{b}$ and a finite set of formulas Δ, there is an infinite convex subset of I which is an indiscernible sequence over $A\bar{b}$ with respect to formulas in Δ.
- For every set A, an A-indiscernible sequence I of order type $|T|^+$ and a set B of cardinality $|T|$, I is eventually indiscernible over $A \cup B$.

Cutting indiscernibles - strong dependence

Strong dependence is in a sense a "global" version of dependence, namely,
Theorem (Shelah) The following are equivalent for a theory T:

- T is strongly dependent.
- For every set A, an infinite A-indiscernible sequence $I = \langle \bar{a}_i : i < \omega \rangle$ (maybe the length of $\bar{a}$ is ω!) and a finite tuple $\bar{b}$, there is an infinite convex subset J of I such that all elements of J have the same type over $A\bar{b}$.
- For every set A, an infinite A-indiscernible sequence $I = \langle \bar{a}_i : i < \omega \rangle$ (maybe the length of $\bar{a}$ is ω!) and a finite tuple $\bar{b}$, there is an infinite convex subset J of I which is an indiscernible sequence over $A\bar{b}$.

Extracting indiscernibles - strong dependence

- *Theorem* (Shelah): Any long enough sequence in a model of a strongly dependent theory has an indiscernible subsequence.

Dp-minimality

- A theory T is *not* dp-minimal if there exist $I = \langle \bar{a}_i \colon i < \omega \rangle$, $J = \langle \bar{b}_i \colon i < \omega \rangle$ and formulas $\varphi(x, \bar{y}), \psi(x, \bar{z})$ (x is a singleton!) such that for every $n, m < \omega$ the set

$$\left\{ [\varphi(x, \bar{a}_i)]^{(n=i)}, \left[\psi(x, \bar{b}_i)\right]^{(m=i)} : i < \omega \right\}$$

  is consistent.
- Again, one can add in addition that I, J are mutually indiscernible and demand only

$$\varphi(x, \bar{a}_0) \wedge \neg\varphi(x, \bar{a}_1) \wedge \psi(x, \bar{b}_0) \wedge \neg\psi(x, \bar{b}_1)$$

  is consistent.

Strong dependence and stability?

- o-minimal, weakly o-minimal, C-minimal, stable U-rank 1 theories are dp-minimal.
- A natural question is: what are stable strongly dependent theories? It is easy to see that a superstable theory is strongly dependent. Are there others?
- In fact, there are: for example, the theory of infinitely many nested equivalence relations (E_{n+1} refines each class of E_n into infinitely many infinite classes) is strongly dependent, and even dp-minimal.
- In order to understand things better, let us look at strong dependence in a slightly different way.

Indiscernible arrays

- We will call an array $\mathfrak{a} = \langle \bar{a}_i^\alpha : \alpha < \kappa, i < \lambda \rangle$ *indiscernible* over a set A if for a fixed $\alpha < \kappa$, the sequence $\bar{a}_{<\lambda}^\alpha$ is indiscernible over $A \cup \bar{a}_{<\lambda}^{\neq\alpha}$. That is, $\mathfrak{a}$ is a collection of sequences which are indiscernible over each other (and over A).
- We will call an array $\mathfrak{a} = \langle \bar{a}_i^\alpha : \alpha < \kappa, i < \lambda \rangle$ *Morley* over a set A if for a fixed $\alpha < \kappa$, the sequence $\bar{a}_{<\lambda}^\alpha$ is based on $(A, A \cup \bar{a}_{<\lambda}^{\neq\alpha})$. That is, $\mathfrak{a}$ is a collection of sequences which are Morley over each other (based on A).
- Let $\mathfrak{a}$ be indiscernible over a set A. Then there exists $B \supseteq A$ such that $\mathfrak{a}$ is Morley over B.

Dividing systems

- A *dividing system* $\mathfrak{Y}$ for a type $p(\bar{x}) \in \mathrm{S}(A)$ consists of
  - an array $\mathfrak{a} = \langle \bar{a}_i^\alpha \colon \alpha < \kappa, i < \omega \rangle$
  - a sequence of formulae $\Phi = \langle \varphi_\alpha(\bar{x}, \bar{y}) \colon \alpha < \kappa \rangle$

  such that

  1. $\mathfrak{a}$ is indiscernible over A.
  2. $p \cup \{\varphi(\bar{x}, \bar{a}_0^\alpha) \colon \alpha < \kappa\}$ is consistent
  3. For every $\alpha < \kappa$, the set
  $$\Sigma_{\mathfrak{Y},\alpha} = \{\varphi_\alpha(\bar{x}, \bar{a}_i^\alpha) \colon i < \lambda\}$$
     is inconsistent.

- We call κ in the definition above the *depth* of $\mathfrak{Y}$.
- A dividing system $\mathfrak{Y} = (\mathfrak{a}, \Phi)$ for p is called *Morley* if $\mathfrak{a}$ is Morley.

Dividing weight

- We say that the (dividing) *pre-weight* of a type p is at least μ (where μ is an ordinal) if for every $\kappa < \mu$ there exists a Morley dividing system $\mathfrak{Y}$ for p of depth κ. The pre-weight of a type p, $\mathrm{pwt}(p)$ is the supremum (if exists) of the depths of Morley dividing systems for p. If the supremum does not exist, we say that p has *unbounded* pre-weight.
- The weight of a type p, $\mathrm{wt}(p)$ is the supremum over all nonforking extensions q of p of $\mathrm{pwt}(q)$ (could be unbounded).
- We say that a type p has *rudimentarily finite pre-weight* if there is no Morley dividing system for p of depth ω. We say that a type p has *rudimentarily finite weight* if every nonforking extension of it has rudimentarily finite pre-weight.

Dividing weight = weight

- In a stable theory, the notions defined above agree with the classical ones:
- Dividing = forking in a stable (and even simple) theory.
- "Kim's Lemma": if $\varphi(x, a)$ divides over a set A, then every Morley sequence in $\mathrm{tp}(a/A)$ exemplifies this.

Strong theories

- Note that theories with bounded dividing weight are precisely NTP_2 and theories with rudimentarily finite weight are precisely "strong" theories.
- This is because, although burden $\neq$ weight, the difference is not significant. More precisely, a dividing system can be easily turned into a Morley deciding system over a bigger set of parameters.
- So where does strong dependence come in?

Randomness/independence systems

- A *randomness (independence) system* $\mathfrak{X}$ for a type $p(\bar{x}) \in \mathrm{S}(A)$ consists of
  - an array $\mathfrak{a} = \langle \bar{a}^\alpha_i : \alpha < \kappa, i < \lambda \rangle$ (where λ,κ are ordinals, λ is infinite)
  - a sequence of formulae $\Phi = \langle \varphi_\alpha(\bar{x}, \bar{y}) : \alpha < \kappa \rangle$

  such that

  1. $\mathfrak{a}$ is indiscernible over A.
  2. For every $\eta \in {}^\kappa\lambda$, the set

  $$\Sigma_{\mathfrak{a},\eta} = \{\varphi_\alpha(\bar{x}, \bar{a}^\alpha_{\eta(\alpha)}) : \alpha < \kappa\} \cup \{\neg\varphi_\alpha(\bar{x}, \bar{a}^\alpha_i) : \alpha < \kappa, i \neq \eta(\alpha)\}$$

  is consistent with $p(\bar{x})$.
- We call κ in the definition above the *depth* of $\mathfrak{Y}$.
- A randomness pattern $\mathfrak{X} = (\mathfrak{a}, \Phi)$ for p is called *Morley* if $\mathfrak{a}$ is Morley.

From dividing to independence

- *Exercise.* Let $p(x)$ be a type over a set A, $n < \omega$ and let $\langle b_i^\alpha \colon \alpha < n, i < \omega \rangle$, $\{\varphi_\alpha(x, y_\alpha) \colon \alpha < n\}$ be a dividing pattern for p over A of depth n. Then there exists a randomness pattern for p over A of depth n; in fact, the randomness pattern is given by the same array and collection of formulae.

From independence to dividing

- *Observation.* Let T be dependent. If there exists a (Morley) randomness system $\mathfrak{X} = (\mathfrak{a}, \Phi)$ for a type p, then there exists a (Morley) dividing system $\mathfrak{X}' = (\mathfrak{a}', \Phi')$ for p.

- *Proof.* Take $\Phi' = \langle \varphi'_\alpha(\bar{x}, \bar{y}_1^\alpha {}^\frown \bar{y}_2^\alpha) \colon \alpha < \kappa \rangle$ where $\varphi'_\alpha(\bar{x}, \bar{y}_1^\alpha {}^\frown \bar{y}_2^\alpha) = \varphi_\alpha(\bar{x}, \bar{y}_1^\alpha) \wedge \neg\varphi_\alpha(\bar{x}, \bar{y}_2^\alpha)$ and let $\mathfrak{a}' = \{\bar{a}_{2i}^\alpha \bar{a}_{2i+1}^\alpha \colon \alpha < \kappa, i < \lambda\}$. It is easy to check that this is still a randomness pattern. It is dividing since T is dependent, and therefore the set

$$\{\varphi(\bar{x}, \bar{a}_i^\alpha)^{\text{parity}(i)} \colon i < \lambda\}$$

  can not be consistent for any α. Clearly, if the original pattern was Morley, so is the new one.

So

- T is strongly dependent if and only if every type in finitely many variables has rudimentarily finite (dividing) pre-weight if and only if every type in finitely many variables has rudimentarily finite (dividing) weight.

- If T is dependent, then every type has bounded pre-weight (and weight).

- Similarly, a theory is dp-minimal if and only if every 1-type has weight 1.

Strongly stable theories

- T is strongly dependent and stable (called *strongly stable*) if and only if every type in finitely many variables has rudimentarily finite weight.

- Hence in a strongly stable theory every type is domination equivalent to a free product of types of weight 1 (not necessarily regular).

- Lachlan's Theorem is true for strongly stable theories, namely: a countable strongly stable theory has either 1 or infinitely many countable models.

- Similarly, a stable theory is dp-minimal if and only if every 1-type has weight 1.

4 Motivations

Motivations for further questions.

- It makes sense to replace an element with a Morley sequence in a dependent theory. Still, one wonders whether this can be avoided. Clearly, this relates to always being able to construct "nice" mutually indiscernible sequences starting with b_i, which in turn relates to notions of "orthogonality".

- Possibly this has to do with the set $\{b_i\}$ being independent in some strong sense. For example, recall:
- In a stable theory, $a \mathop{\smile\!\!\!\!\!\!|}\, b$ if and only if for every I, J starting with a, b respectively, there are I', J' such that
  1. $I' \equiv_a I, J' \equiv_b J$
  2. I', J' are mutually indiscernible.
- For the purpose of this talk, we will call a, b satisfying (1) ,(2) above *strictly independent*.

Motivation I - thorn-weight

(joint with Alf Onshuus).

Let T be rosy and strongly dependent (or even strong).

- Every type has rudimentarily finite thorn-weight.
- Hyttinnen's Lemma is true for thorn-forking, hence every type has *finite* thorn-weight, and is, in fact, thorn-domination equivalent to a product of finitely many weight-1 types.

Rudimentarily finite thorn-weight

How does one show that every type has rudimentarily finite thorn weight?

- *Lemma*: Let $\{a_i\}_{i<\alpha}$ be thorn-independent. Then there are mutually indiscernible sequences I_i starting with a_i. That is, there are I_i such that
  - I_i starts with a_i
  - I_i is indiscernible over $I_{\neq i}$
- Work with *strong dividing* and remember that
  - If $\varphi(x, a)$ strongly divides (over $\varnothing$, say) then *every* infinite indiscernible sequence in $\text{tp}(a)$ witnesses dividing of $\varphi(x, a)$.

Motivation II - generically stable weight

- We define *generically stable weight* of a type p as follows:

  Let $p(x)$ be any type over some model M. We say that $a, \langle b_i\rangle_{i=1}^{\alpha}$ witnesses (*pre-weight of* p is at least α) if $a \models p(x)$, $\langle b_i\rangle_{i=1}^{\alpha}$ is an M-independent set, $\text{tp}(b_i/M)$ is generically stable, and $a \not\mathop{\smile\!\!\!\!\!\!|}_M b_i$ for all i.
- One defines pre-weight, weight, rudimentarily finite (generically stable) weight as usual.
- This is a natural attempt to "isolate" and understand the "stable" part of a type.

Rudimentarily finite generically stable weight

- *Lemma.* Let $\langle b_i \rangle_{i=1}^{\alpha}$ be an M-independent set of generically stable elements, and I_i is an M-indiscernible sequence starting with b_i. Then there are I'_i starting with b_i such that
  - $I'_i \equiv_M I_i$
  - I'_i is $MI'_{\neq i}$-indiscernible
- *Corollary.* Every type in a strongly dependent theory has rudimentarily finite generically stable weight.

From rudimentarily finite to finite?

- *Question.* Is it true that any type in a strongly dependent theory has *finite* generically stable weight?
- *Theorem.* Every generically stable type in a strongly dependent theory has rudimentarily finite generically stable weight (which equals to its dividing weight).
- The proof requires some new techniques, because it is not true that a forking increasing chain of gen. stab. types needs to be bounded.

Question

- What can be shown in general (that is, without assuming that b_i are generically stable)?
- For example, what can be said about "strict independence"?
- What if we only require that given I, J starting with a, b there are J', I' of the same type (and starting with the same elements) such that e.g. I is indiscernible over J, and J is indiscernible over a?
- What if we only require I indisc. over b and J - over a?
- Note that all of the above are equivalent to $a \mathop{\smile\!\!\!\!\!\!|}\, b$ in a stable theory.

Some unsatisfactory answers

- *Lemma1 (Onshuus, U.)* If $\{a_\alpha \colon \alpha < \lambda\}$ is a nonforking sequence, then there are mutually indiscernible J_α starting with a_α (but one can not control their type!).
- *Lemma2.* If $\{a_\alpha \colon \alpha < \lambda\}$ is a nonforking set, then whenever there are indiscernible sequences J_α starting with a_α there are indiscernible sequences J'_α starting with *the same* a_α such that J'_α is indiscernible over $a_{\neq \alpha}$ and $J'_\alpha \equiv J_\alpha$.
- In fact, if we work over a model, this is an equivalence (because forking implies dividing).
- All these (and other similar results) use boundedness of nonforking.

5 Strict nonforking

- Strict nonforking.

Strict nondividing

- Let $A \subseteq B$. We say that a type $p \in \mathrm{S}(B)$ is a strictly nondividing extension of $p\restriction A$ if for every $a \models p$
  - $\mathrm{tp}(a/B)$ does not divide over A
  - $\mathrm{tp}(B/Aa)$ does not divide over A.
- We will say that a type $p \in \mathrm{S}(B)$ *co-divides* over a set A if there is $a \models p$ such that $\mathrm{tp}(B/Aa)$ divides over A. In other words, p co-divides over A if there exist $a \models p$ and a formula $\varphi(x, b) \in p$, such that $\varphi(a, y)$ divides over A.
- So $p \in \mathrm{S}(B)$ is a strictly non-dividing extension over A if and only if it does not divide and does not co-divide over A.

Strict nonforking

- Let $A \subseteq B$. We say that a type $p \in \mathrm{S}(B)$ is a *strictly nonforking* (or strictly free) extension of $p\restriction A$ if there exists a global type q extending p which is a strictly nondividing extension of $p\restriction A$.
- We also say that p is *strictly nonforking* over A. If $a \models p$, we write $a \mathop{\smile\kern-0.9em\raise0.4ex\hbox{$|$}}^{st}_A B$.

Strictly nonforking extensions

Let N be saturated enough over A. Then

- A type $p \in \mathrm{S}(N)$ is strictly nonforking over A if and only if for every $a \models p$
  - $a \mathop{\smile\kern-0.9em\raise0.4ex\hbox{$|$}}_A N$
  - $\mathrm{tp}(N/Aa)$ does not divide over A.
- If $p \in \mathrm{S}(N)$ is a heir of $p\restriction A$ and does not fork over A, it is strictly nonforking over A. In particular this is the case if p is both a heir and a co-heir of $p\restriction A$.

Strict Morley sequences

- Let O a linear order, A a set. We call a sequence $I = \langle a_i : i \in O \rangle$ a *strict Morley sequence over B based on A* if it is an indiscernible sequence over B and $\mathrm{tp}(a_i/Ba_{<i})$ is strictly free over A for all $i \in O$.
- In the previous definition, we omit "based on A" if $A = B$.
- Let $p \in \mathrm{S}(B)$ be a type. We call a sequence I a *strict Morley sequence in p* if it is a strict Morley sequence over B of realizations of p.

Strict Morley sequences and dividing

"Kim's Lemma" for dependent theories.

- Assume A is an extension base.

  Let $\varphi(x,a)$ be a formula which divides over a set A. Then every strict Morley sequence I in $\mathrm{tp}(a/A)$ witnesses dividing; that is, the set $\varphi(x,I) = \{\varphi(x,a') : a' \in I\}$ is inconsistent.

- Existence of strict Morley sequences (e.g. over models) follows from the work of Kaplan and Chernikov. In fact, using their results one can show that every type over a model has a global nonforking heir.

Properties of strong nonforking

(with Itay Kaplan)

Let M be a model.

- Strong nonforking over M is symmetric.
- The following are equivalent for $p \in \mathrm{S}(M)$
  - Strong nonforking satisfies transitivity on the set of realizations of p.
  - Strong nonforking coincides with nonforking for realizations of p.
  - p is generically stable.

Proof of symmetry

Lemma. Assume $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^{st} a$, then for any c, there is some $c_0 \equiv_{Mb} c$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M bc_0$ and $bc_0 \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M a$.

Proof. Let $p(x) = \mathrm{tp}(c/Mb)$. We want the following set to be consistent with $p(x)$:

$$\{\neg\varphi(x,b,a) : \varphi(x,b,a) \text{ forks over } M\} \cup$$
$$\cup \{\neg\varphi(a,b,x) : \varphi(y,b,c) \text{ forks over } M\}$$

Suppose not. By forking = dividing over models, we have that

$$p(x) \vdash \varphi_1(x,b,a) \bigwedge \varphi_2(a,b,x)$$

where $\varphi_1(x,y,a)$ divides over M, and $\varphi_2(y,b,c)$ divides over M.

Proof of Symmetry II

Let I be an indiscernible sequence witnessing $\varphi_1(x,y,a)$ divides, wlog $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^{st} I$. So I is indiscernible over Mb. Hence for some $m < \omega$,

$$p(x) \vdash \bigvee_{i<m} \varphi_2(a_i,b,x)$$

.

Recall that also $I \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M b$, so $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M b$. Now let $c_0 \equiv_{Mb} c$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M bc_0$, so for some i, $\varphi_2(a_i,b,c_0)$, which is a contradiction ($\varphi_2(y,b,c)$ divides, hence forks).

Characterization of strict independence

Theorem (Kaplan, U.) The following are equivalent for a model M and elements a, b

1. $a \mathop{\smile\!\!\!\!\!\mid}_M^{st} b$
2. $b \mathop{\smile\!\!\!\!\!\mid}_M^{st} a$
3. a and b are strictly independent over M, that is, for every I, J starting with a, b respectively, there are I', J' starting with a, b of the same type (over M) as I, J, such that I', J' are mutually indiscernible over M.
4. For every I, J starting with a, b respectively, there are I', J' starting with a, b of the same type (over M) as I, J, such that I' is indiscernible over J' and J' is indiscernible over a.
5. There are $a \in N_a, b \in N_b$ containing M, saturated in $|M|^+$ such that $N_a \mathop{\smile\!\!\!\!\!\mid}_M N_b$ and $N_b \mathop{\smile\!\!\!\!\!\mid}_M N_a$

Characterization of strict nondividing

Recall: The following are equivalent for a model M and elements a, b

1. $\mathrm{tp}(a/Mb)$ is strictly nondividing over M
2. $\mathrm{tp}(b/Ma)$ is strictly nondividing over M
3. $a \mathop{\smile\!\!\!\!\!\mid}_M b$ and $b \mathop{\smile\!\!\!\!\!\mid}_M a$
4. For every I, J starting with a, b respectively, there are I', J' starting with a, b of the same type (over M) as I, J, such that I' is indiscernible over b and J' is indiscernible over a.

Question: is this equivalent to strict independence?

Weak "Local character" for dependent theories

- Let M be a model. Let $\langle a_\alpha : \alpha < \lambda \rangle$ be a strongly nonforking sequence (that is, $a_\alpha \mathop{\smile\!\!\!\!\!\mid}_M a_{<\alpha}$), b an element. Then for almost all (except $|T|$-many) α we have $b \mathop{\smile\!\!\!\!\!\mid}_M^{st} a_\alpha$, and even $b \mathop{\smile\!\!\!\!\!\mid}_M^{st} a_\alpha$.
- Finally, we can define weight based on strict nonforking. Then we obtain the following quite desirable properties:
- In every dependent theory, a type over a model has bounded pre-weight. A dependent theory is strongly dependent if and only if every type over a model has almost finite pre-weight.

Weight - more precisely

- Let $p(x)$ be any type over a model M. We say that $a, \langle b_i \rangle_{i=1}^{\alpha}$ witnesses *strict pre-weight of p is at least α* if $a \models p(x)$, $b_i \mathop{\smile\!\!\!\!\!\mid}_M^{st} b_{<i}$ for all $i < \alpha$, and $a \not\mathop{\smile\!\!\!\!\!\mid}_M b_i$ for all i.
- The *strict pre-weight* of p is the supremum over all α such that such a witness exists.
- Note: in the definition above, one can replace $a \not\mathop{\smile\!\!\!\!\!\mid}_M b_i$ with $a \not\mathop{\smile\!\!\!\!\!\mid}_M^{st} b_i$. Gives rise to a different notion,but the properties below stay true.
- In every dependent theory, a type over a model has bounded strict pre-weight. A dependent theory is strongly dependent if and only if every type over a model has almost finite strict pre-weight.

Different notions of orthogonality

- (Shelah) We call two types $p, q \in \mathrm{S}(A)$ *weakly orthogonal* or if $p(x) \cup q(y)$ is a complete type over A. We write $p \perp_w q$. If a, b realize p, q respectively, then we write $a/A \perp_w b/A$ or $a \perp_w b$ when A is fixed and clear from the context.
- We call $\mathrm{tp}(a/A), \mathrm{tp}(b/A)$ *weakly orthogonal*[1] if whenever I, J are A-indiscernible sequences starting with a, b respectively, there are I', J' mutually A-indiscernible such that $I \equiv_{Aa} I'$ and $J \equiv_{Ab} J'$. We write $a/A \perp_w^1 b/A$
- Let A be an extension base(e.g. a model), $p, q \in \mathrm{S}(A)$. If $p \perp_w q$, then $p \perp_w^1 q$.

Different **notions of orthogonality**

- Let $(\mathbb{Q}, <, P)$ be the theory of $(\mathbb{Q}, <)$ with a dense co-dense predicate P. Let $p, q \in \mathrm{S}(\mathbb{Q})$ be the types over the prime model $\mathbb{Q}$ such that if a, b realize p, q respectively, then $a, b > \mathbb{Q}$, $P(a), \neg P(b)$.

  Clearly $p \not\perp_w q$ since p, q do not determine whether $a < b$ or $b < a$. On the other hand, it is easy to see that $p \perp_w^1 q$. It is also the case that $p \perp_w^{st} q$.
- So we have two different reasonable notions of orthogonality. Of course, in stable theories they coincide.