A Combinatorial Study of Linear Deterministic Relay Networks

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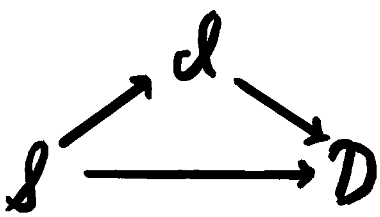
Relay Network:

Source $\mathcal{S}$, Destination $\mathcal{D}$,

one or more intermediate nodes to help

Ex:



$\mathcal{S}$: transmits x

$\mathcal{R}$: receives y_1,
transmits x_1

$\mathcal{D}$: receives y

Description: $P(y, y_1 \mid x, x_1)$

Capacity generally unknown

even for Gaussian relay network:

$$\left.\begin{aligned} Y_1 &= X + Z_1 \\ Y &= X + Z_1 + X_1 + Z_2 \end{aligned}\right\} \begin{array}{l} Z_1, Z_2 \text{ independent} \\ \text{Gaussian r.v.'s} \end{array}$$

Important for study of wireless commun.

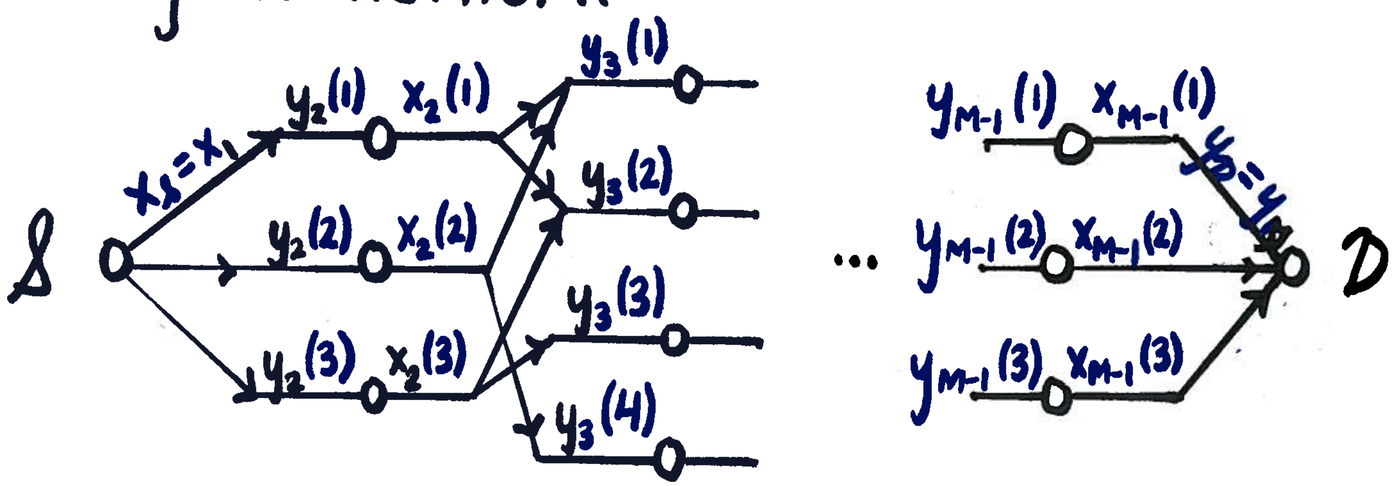
Avestimehr, Diggavi, Tse (2007):

New relay network model:

- to approximate the capacity of Gaussian relay networks

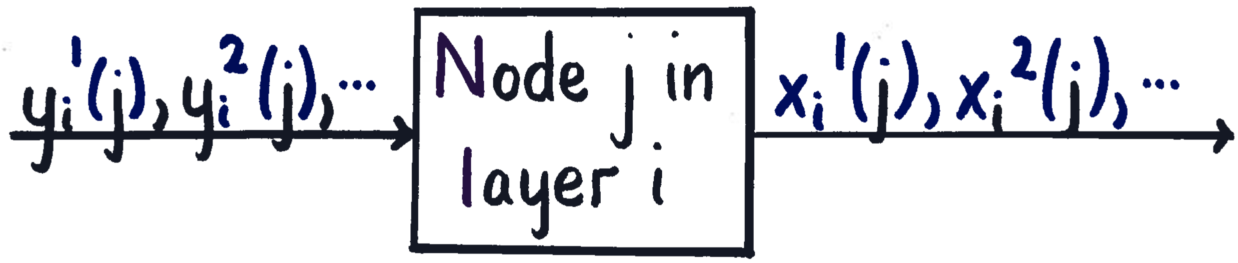
- to design near-optimal coding schemes for them

Layered network:



Nodes transmit and/or receive vectors over finite field $\mathbb{F}_q$

m_i nodes in layer i : $m_1 = m_M = 1$



$x_i^t(j)$ can be an arbitrary function of the vectors received at the node before $x_i^t(j)$ is transmitted.

Received vectors:

Given matrices G_{i-1} over $\mathbb{F}_q$:

$$y_i^t(k) = \sum_{j=1}^{m_{i-1}} G_{i-1}(k,j)\, x_{i-1}^t(j), \quad 1 \le k \le m_i$$

Block matrix $G_{i-1} = [G_{i-1}(k,j)]$

Avestimehr, Diggavi, Tse:

Generalization of traditional networks

- Time expansion

Key Questions: · Capacity?

· How to optimize coding?

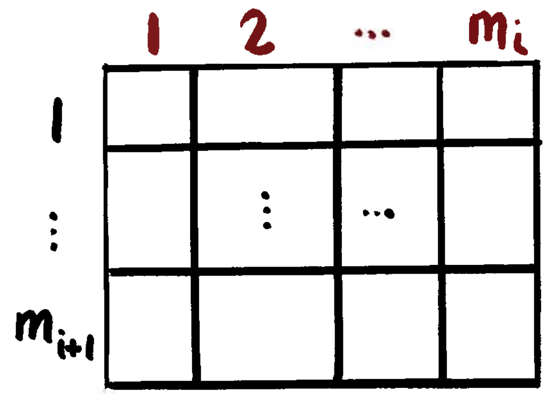
· Is network coding needed?

CAPACITY $\mathcal{C}$:

Recall block matrix

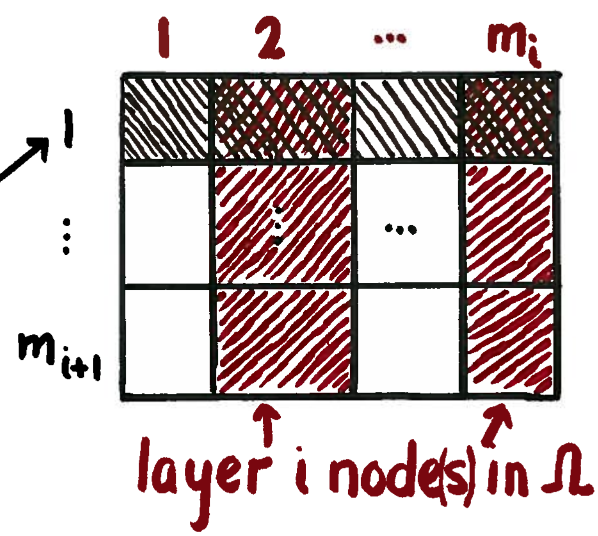
$G_i = [G_i(k,j)]$, $1 \le k \le m_{i+1}$, $1 \le j \le m_i$ describes

layer i $\longrightarrow$ layer $i+1$: $y_{i+1} = G_i x_i$



Cut Ω: A subset of nodes

$G_i(\Omega)$: Block submatrix of G_i: layer $i+1$ node(s) in Ω



$$\mathcal{C}(\Omega) = \sum_{i=1}^{M-1} \operatorname{rank}(G_i(\Omega))$$

Thm (A-D-T): $\displaystyle \mathcal{C} = \min_{\Omega \text{ separates } \mathcal{S} \text{ and } \mathcal{D}} \mathcal{C}(\Omega)$

"Max-flow / min-cut" result

Upper Bound: Information-theoretic cut set

Achievability: - Random Coding

- Each $x_i^t(j)$ is a function of $y_i^1(j), y_i^2(j), \cdots, y_i^\tau(j)$ for some large τ: Delay, Complexity
- Not a "flow"

Amaudruz and Fragouli (2009):

- $q=2$
- Deterministic, polynomial-time algorithm, $\tau=1$
- Path augmentation, flow-based scheme
- How to generalize for $q>2$?

Our Contributions:

- Deterministic, polynomial-time algorithm
- Arbitrary finite field
- $x_i^t(j)$ depends only on $y_i^t(j)$ and in a simple way
- Flow-based scheme
- New connections to matroid theory and combinatorial optimization

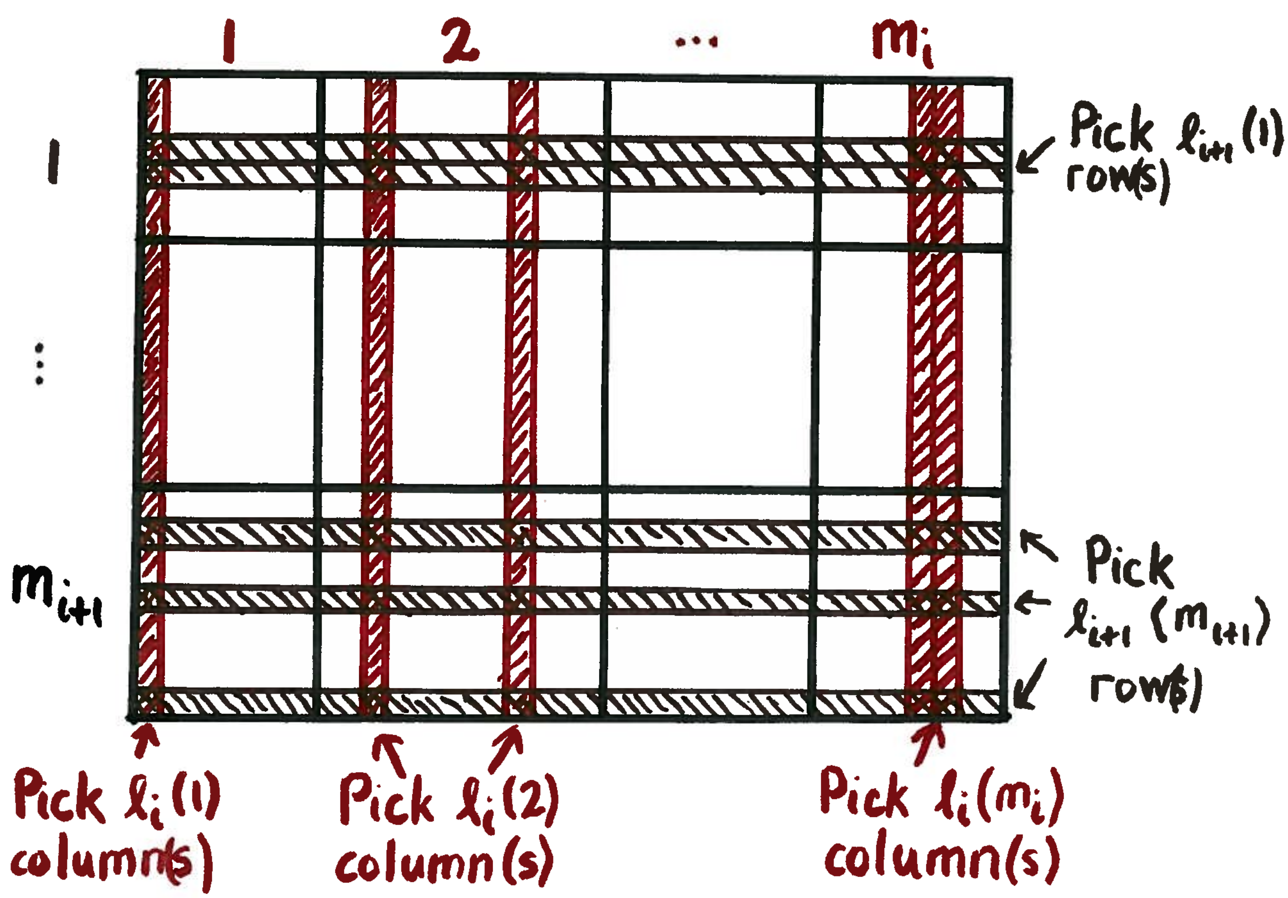
Overview:

- Flow from layer i nodes to layer i+1 nodes
- Flow over a network
- Approach to results

Flow from Layer i to Layer i+1:

Suppose: · there are positive integers $\ell_i(1), \cdots, \ell_i(m_i)$ and $\ell_{i+1}(1), \cdots, \ell_{i+1}(m_{i+1})$ with $R = \sum_{j=1}^{m_i} \ell_i(j) = \sum_{k=1}^{m_{i+1}} \ell_{i+1}(k)$, and

· there is a full-rank submatrix G_{d_i} of block matrix G_i formed by



Coding Scheme at Layer i

Let $x_i = \begin{bmatrix} x_i(1) \\ \vdots \\ x_i(m_i) \end{bmatrix}$, $y_{i+1} = \begin{bmatrix} y_{i+1}(1) \\ \vdots \\ y_{i+1}(m_{i+1}) \end{bmatrix}$

Recall that $y_{i+1} = G_i \, x_i$

Let $x_{d_i}(j) = \ell_i(j)$ components of $x_i(j)$ corresponding to the column chosen for G_{d_i}

Set the remaining components of $x_i(j)$ to 0.

Let $y_{d_{i+1}}(k) = \ell_{i+1}(k)$ components of $y_{i+1}(k)$ corresponding to the rows chosen for G_{d_i}

Then $x_{d_i} = \begin{pmatrix} x_{d_i}(1) \\ \vdots \\ x_{d_i}(R) \end{pmatrix} = G_{d_i}^{-1} \, y_{d_{i+1}} = G_{d_i}^{-1} \begin{pmatrix} y_{d_{i+1}}(1) \\ \vdots \\ y_{d_{i+1}}(R) \end{pmatrix}$

$\Rightarrow$ R units of information "flow" from layer i to i+1.

Question: What are the conditions on G_i that enable the construction of G_{d_i} with the desired properties?

Transversal theory:

Study of systems of representatives

Rado-Hall Theorem (1942):

Let (E,r) be a matroid and $A_1,\cdots,A_n \subseteq E$. Given non-negative integers $\ell_1,\cdots,\ell_n$ there exist disjoint subsets $a_1 \subseteq A_1, a_2 \subseteq A_2, \cdots, a_n \subseteq A_n$ with $|a_i| = \ell_i$ and $a_1 \cup a_2 \cup \cdots \cup a_n$ an independent set iff for every subset $I \subseteq \{1,2,\cdots,n\}$:

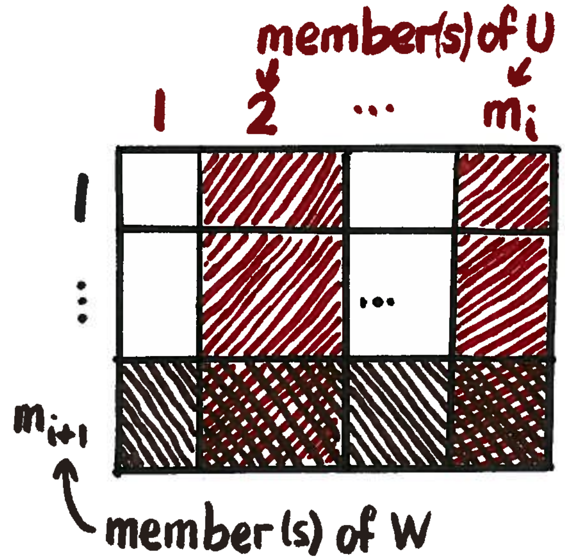
$$r\left(\bigcup_{i\in I} A_i\right) \geq \sum_{i\in I} \ell_i .$$

For our problem we need a new application of the Rado-Hall theorem:

For $U \subseteq \{1, \cdots, m_i\}$,

$W \subseteq \{1, \cdots, m_{i+1}\}$,

form $G_i(W, U) \Rightarrow$



Theorem: We can obtain the desired G_{d_i} iff for all choices of U and W,

$$\mathrm{rank}(G_i(W,U)) \geq \sum_{j \in U} \ell_i(j) + \sum_{k \in W} \ell_{i+1}(k) - R$$

Proof uses a variation of "bimatroids" Kung (1978), Schrijver (1978).

Note: Original max-flow/min-cut thm (FF 1956) related to Hall's marriage thm (1935).

Flow over the Network:

Suppose: · there are non-negative integers $\ell_2(1), \cdots, \ell_2(m_2)$

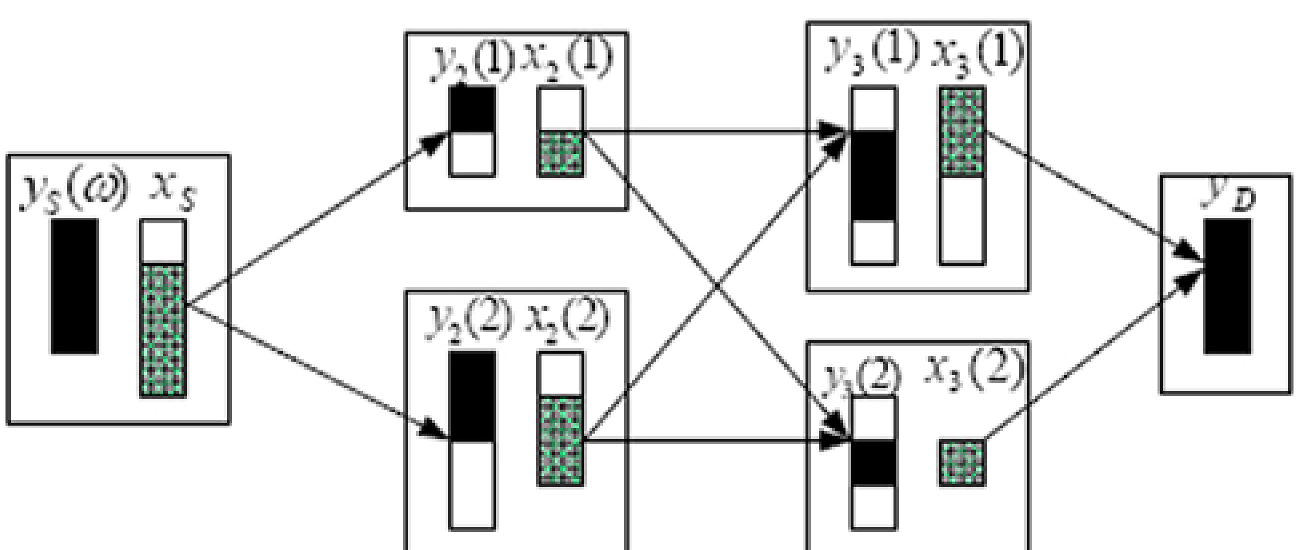
$$\ell_3(1), \cdots, \ell_3(m_3)$$

$$\vdots$$

$$\ell_{M-1}(1), \cdots, \ell_{M-1}(m_{M-1})$$

with $R = \ell_1(1) = \ell_M(1) = \sum_{j=1}^{m_i} \ell_i(j)$ for each i

· there is a rank-R square submatrix G_{d_i} of G_i for a flow scheme from layer i to layer $i+1$ for each i



Coding Scheme for the Network

- **At the source:**
  - Encode the R units of information into the components of $x_s = x_1$ corresponding to the columns of G_1 chosen for G_{d_1}.
  - Set the remaining components of x_1 to 0.

- **At node j of layer i, $2 \leq i \leq M-1$:**
  - Set $y_{d_i}(j)$ to the $\ell_i(j)$ components of $y_i(j)$ matching the rows of G_{i-1} used for $G_{d_{i-1}}$.
  - Set $x_{d_i}(j) = y_{d_i}(j)$ for the components of $x_i(j)$ corresponding to the columns of G_i selected for G_{d_i}
  - Set the remaining components of x_i to 0.

- At the destination:
  - Set y_{d_M} to be the R components of $y_D = y_M$ corresponding to the rows chosen from G_{M-1} for $G_{d_{M-1}}$.
  - Recover the R units of information transmitted from the source by calculating

$$G_{d_1}^{-1} G_{d_2}^{-1} \cdots G_{d_{M-1}}^{-1} y_{d_M}.$$

Does this coding scheme involve network coding? Yes and No.

Question: What are the conditions on $G_1, G_2, \cdots, G_{M-1}$ that enable the construction of a network flow with the desired properties?

Theorem: We can obtain a network flow as defined above iff $R \leq \mathcal{C}$.

One approach to the proof and part of the algorithm:

Submodular flow [Edmonds and Giles (1977)]:

Classical flow conservation constraints are replaced by submodular flow constraints on certain subsets of nodes.

Our more direct approach to the proof and a polynomial-time algorithm:

$C(\Omega)$ is submodular in Ω

- Use submodular minimization algorithms [Iwata et al. (2000)], [Schrijver (2000) to obtain $\mathcal{C}$.

- Use the polymatroid intersection theorem [Edmonds (1970)] to obtain $\{\ell_i(j)\}$.

- Use a greedy algorithm to find G_{d_i} for each i.

Current and Future Work

- Multicast problem
- Submodular flow
  - Which algorithms lead to the best algorithms for Gaussian relay networks?
  - Extend ideas to k-unicast sessions and see if ideas from approximation algorithms can help construct codes.
  - Other Network IT Problems