Binary matroid minors

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Graph Minors [Robertson & Seymour]

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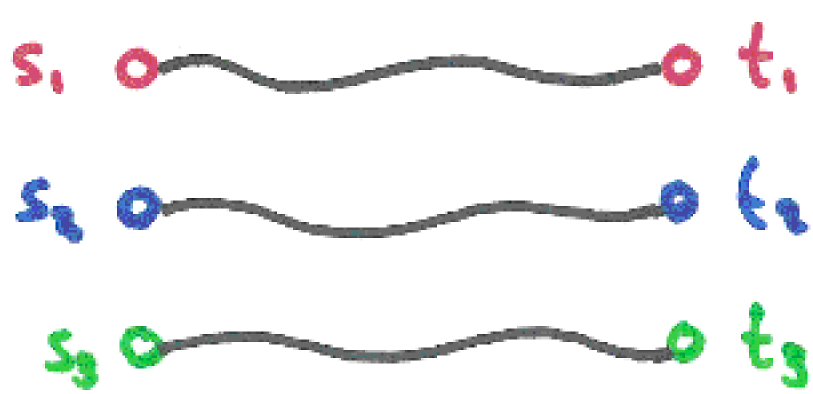
k-Disjoint Paths Algorithm $O(n^3)$



Binary Matroid Minors

WQO Theorem Every minor-closed class of binary matroids is characterized by a finite set of excluded minors.

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Rooted-Minor Testing Algorithm $O(n^7)$

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That is: For any binary matroid N there exist $k, t \in \mathbb{Z}$ such that, if M is a k-connected binary matroid with no N-minor, then there is a rank $\leq t$ perturbation M' of M such that M' or M'^* is graphic.

Weak Structure Theorem. I.

If M is a sufficiently connected binary matroid with no N-minor, then M or M^* is almost graphic.

Weak Structure Theorem. II.

If M is a sufficiently connected binary matroid with no $M(G)^*$-minor, then M is almost graphic.

Perturbation of graphic matroids

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Write $P = C \cdot D$ where D has t rows and C has t columns.

$$\begin{array}{cc} & E \\ V & \left[\begin{array}{c|c|c} I & D & I \\ \hline 0 & A & \underbrace{C}_{t} \end{array}\right] \end{array} \Big\} t$$

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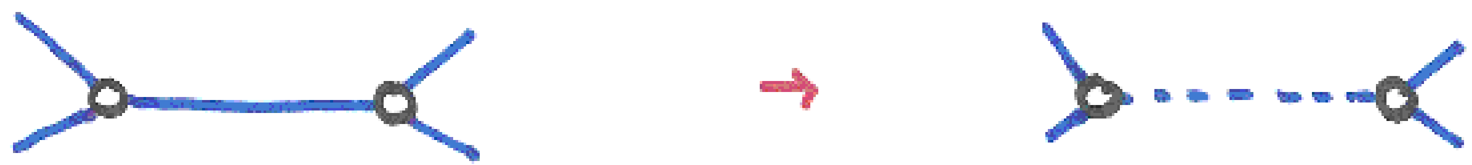
Labelled graphs

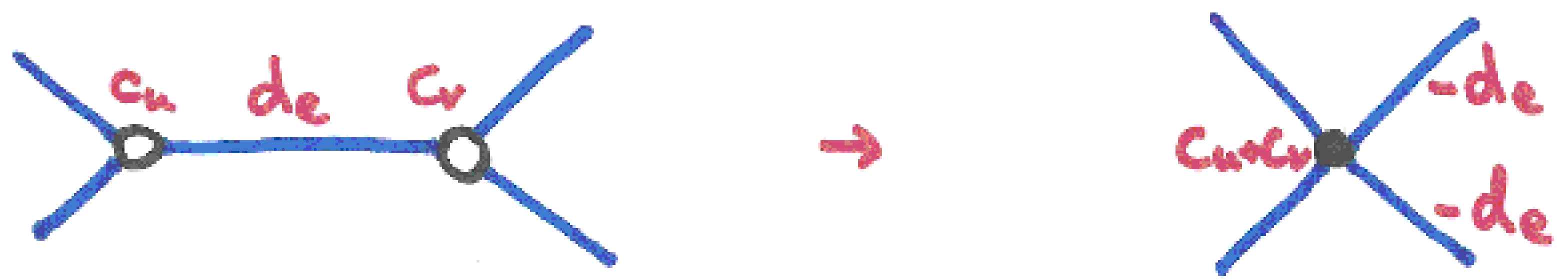
$c_v, d_e \in \{0,1\}^t$

Minors of labelled graphs

deletion



contraction



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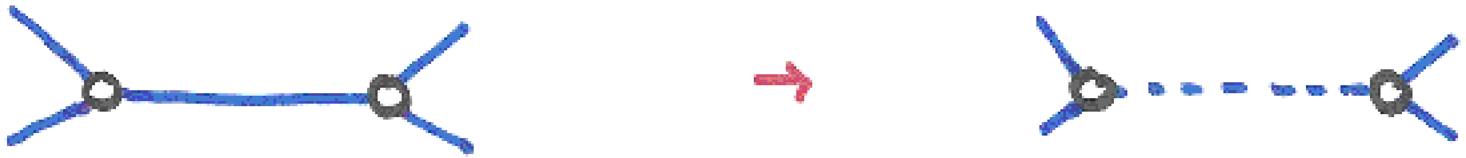

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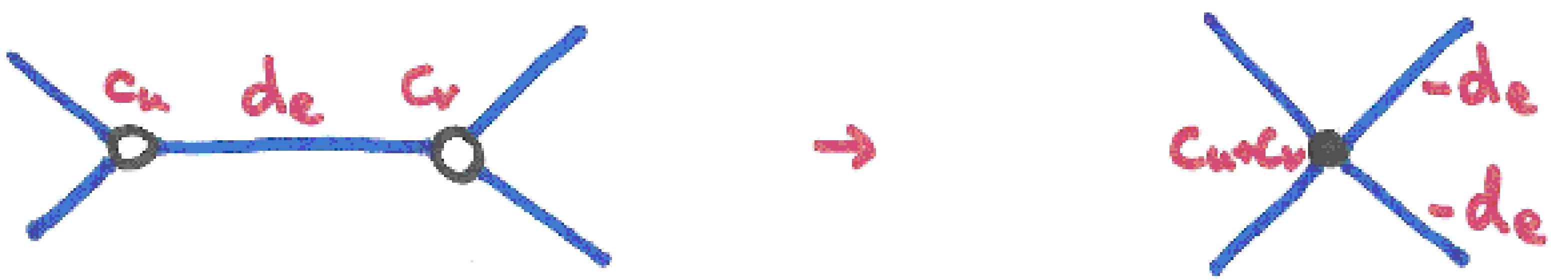

$$\begin{array}{cc} & E \\ V & \left[\begin{array}{c|c|c} I & D & \boxed{} \\ \hline 0 & A & C \end{array}\right] \begin{array}{l} \}\, t \\ \\ \end{array} \\ & \hphantom{I\,D\,}\underbrace{\hphantom{C}}_{t} \end{array}$$

← contraction changes this submatrix

Labelled graph minors

WQO Theorem Every minor-closed class of t-labelled graphs is characterized by a finite set of excluded minors.

H-minor testing algorithm $O(n^3)$

Important minor-closed classes

- branch-width $\leq k$
- cycle matroids of planar graphs
- cycle matroids of apex graphs
- graphic matroids
- rank $\leq t$ perturbations of these classes
- duals of these classes

Conjecture

For any graph G there exists $k \in \mathbb{Z}$ such that, if M is a co-simple binary matroid with no $M(G)$- or $M(G)^*$-minor, then $\mathrm{girth}(M) \leq k$.

Binary codes

$A \in \{0,1\}^{k \times n}$ full row rank

$C(A): w \in \{0,1\}^k \to A^T w \in \{0,1\}^n$

message code word

distance $C(A) :=$ girth $M(A)^*$

Binary codes

$$A \in \{0,1\}^{k \times n} \text{ full row rank}$$

$$C(A): \underbrace{w \in \{0,1\}^k}_{\text{message}} \to \underbrace{A^T w \in \{0,1\}^n}_{\text{code word}}$$

$$\text{distance } C(A) := \text{girth } M(A)^*$$

Theorem [Vardy]
Computing the distance of a binary code is NP-hard.

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message   code word

distance $C(A) :=$ girth $M(A)^*$

Theorem [Vardy]

Computing the distance of a binary code is NP-hard.

Conjecture

Computing the distance of a binary code from a proper minor-closed class is poly-time solvable.

Cycle codes of graphs

$M(A)^*$ is graphic.

Theorem [Folklore]
Maximum likelyhood decoding is poly-time solvable for cycle codes.

Problem For which minor-closed classes of binary codes is the maximum likelyhood decoding problem poly-time solvable?

Channel capacity

imperfect channel: Prob[bit error] = p, $0 < p < 1$.

$C_k : \{0,1\}^k \to \{0,1\}^{n_k}$, $k = 1, 2, 3, \ldots$

$\text{rate}\{C_k\} := \liminf_{k\to\infty} \frac{k}{n_k}$

$\{C_k\}$ is asymptotically error-free if

$$\lim_{k\to\infty} \text{Prob}[\text{transmission error using } C_k] \to 0.$$

Shannon's Theorem For all $0 < p < 1$ and $\varepsilon > 0$, there exist asymptotically error-free sequences of codes with rate $> 1 + p \log_2 p + (1-p) \log_2 (1-p) - \varepsilon$.

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Theorem If $C_k : \{0,1\}^k \to \{0,1\}^{n_k}$ is a sequence of binary codes from a proper minor-closed class and rate $\{C_k\} > 0$, then

$$\lim_{k\to\infty} \frac{\text{distance } C_k}{n_k} = 0.$$

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Theorem [Decreusefond, Zemor] For all $0<p<1$ and $d\in\{3,4,\dots\}$ with $p(1-p) < \dfrac{1}{4(d-1)^2}$ there is an asymptotically error-free sequence of <u>cycle codes</u> with rate $1-\frac{2}{d}$.

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$$(p = 0.001 \Rightarrow R = 0.9885)$$

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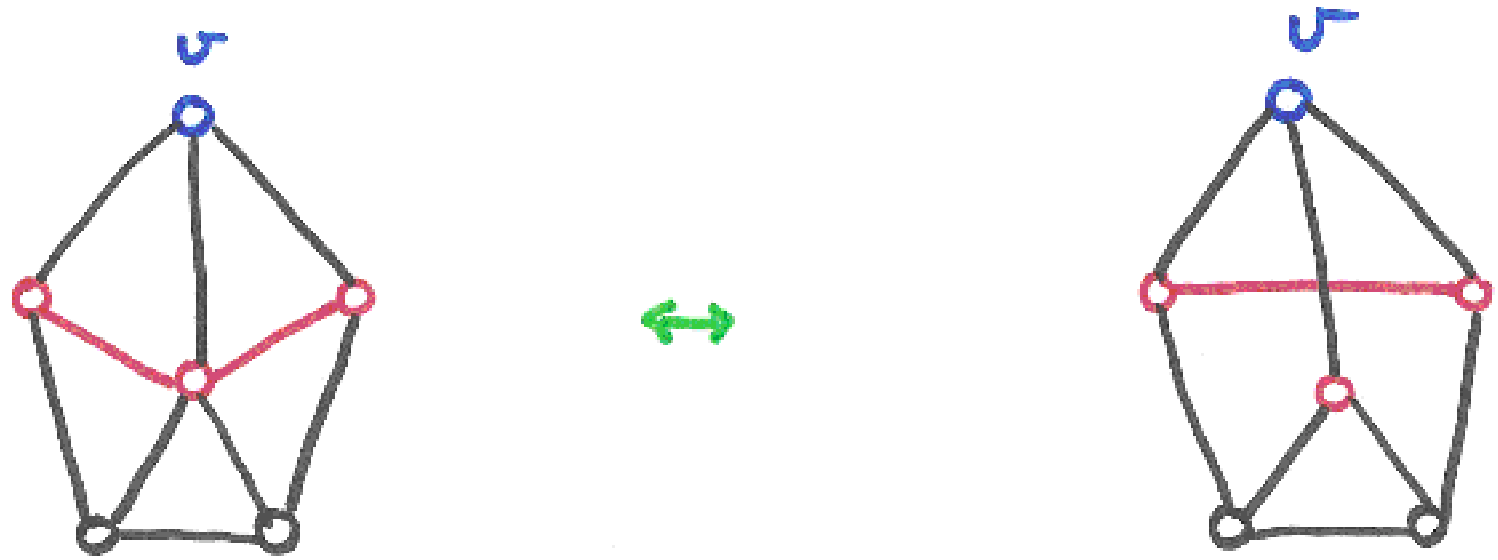
$$(p = 0.001 \Rightarrow R = 0.875)$$

Vertex minors

vertex deletion

local complementation



Vertex-minor-closed classes:

- circle graphs
- rank-width $\leq k$

Open problems

WQO Conjecture Every vertex-minor-closed class is characterized by a finite set of excluded vertex minors.

Vertex-minor testing poly-time?

χ-boundedness conjecture For every proper vertex-minor-closed class $\mathcal{G}$ of graphs, there exists $f: \mathbb{N} \to \mathbb{N}$ such that $\omega(G) \leq \chi(G) \leq f(\omega(G))$ for each $G \in \mathcal{G}$.

Vertex minors and binary matroids

Conjecture For each graph H there is a binary matroid N_H such that, if A is the adjacency matrix of a graph G and $M([I|A])$ has an N_H-minor, then G contains H as a vertex minor.

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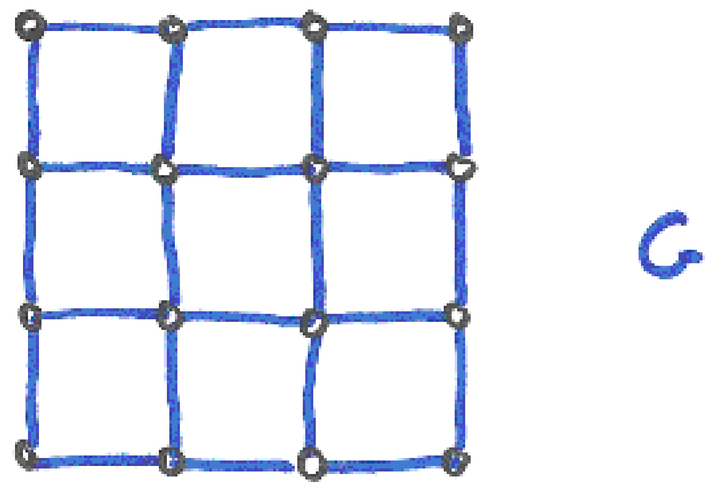
Conjecture For each graph H there is a binary matroid N_H such that, if A is the adjacency matrix of a graph G and $M([I|A])$ has an N_H-minor, then G contains H as a vertex minor.

$\Rightarrow$ structure theorem for vertex-minors.

Measurement-based Quantum Computing

(input state, G) $\rightarrow$ altered state



Conjecture Let $\mathcal{G}$ be a proper vertex-minor-closed class of graphs. Then the measurement-based quantum computations on $\mathcal{G}$ can be efficiently simulated on a classical computer.

Intertwining

M is an <u>intertwine</u> of N_1 and N_2 if M contains N_1 and N_2 as minors but no proper minor of M contains N_1 and N_2.

Theorem For any matroid N there are finitely many intertwines of $U_{2,4}$ and N.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be minor-closed classes.

Note.

(1) $\mathcal{M}_1 \cap \mathcal{M}_2$ and $\mathcal{M}_1 \cup \mathcal{M}_2$ are minor closed.

(2) Each excluded minor of $\mathcal{M}_1 \cap \mathcal{M}_2$ is an excluded minor of $\mathcal{M}_1$ or $\mathcal{M}_2$.

(3) Each excluded minor of $\mathcal{M}_1 \cup \mathcal{M}_2$ is an intertwine of an excluded minor of $\mathcal{M}_1$ and an excluded minor of $\mathcal{M}_2$.

Theorem If $\mathcal{M}_1$ is a class of binary matroids and $\mathcal{M}_2$ has a finite set of excluded minors, then $\mathcal{M}_1 \cup \mathcal{M}_2$ has a finite set of excluded minors.

Beyond binary matroids

Let $\mathbb{F}$ be a finite field.

WQO Conjecture. Every minor-closed class of $\mathbb{F}$-representable matroids is characterized by a finite list of excluded minors.

Minor Testing. Polynomial-time?

Rota's Conjecture

The class of $\mathbb{F}$-representable matroids is characterized by a finite list of excluded minors.

Equivalence Testing Conjecture

There is a poly-time algorithm for testing $M(A_1) = M(A_2)$ for $A_1, A_2 \in \mathbb{F}^{k \times n}$.