# Connections between Minimum Rank and Minimum Semidefinite Rank

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September 20, 2009 – September 26, 2009

Let $G$ be a (simple, undirected, finite) graph, denote the order of $G$ by $|G|$, and let $S_n$ denote the set of real symmetric $n \times n$ matrices. We use the notation, $G(A)$, to describe *the graph of* $A$, and by this we mean the graph on vertices $\{1, 2, \ldots, n\}$ and with $ij$ an edge of $G(A)$ if and only if $i \neq j$ and $a_{ij} \neq 0$. The *minimum rank* of $G$ is

$$mr(G) = \min\{rank(A) : \ A \in S_n \text{ and } G(A) = G\}.$$

The *maximum nullity* of a graph $G$ (over $\mathbf{R}$) is defined to be

$$M(G) = \max\{dim\,(ker(A)) : \ A \in S_n \text{ and } G(A) = G\}.$$

Clearly,

$$mr(G) + M(G) = |G|.$$

Two other families of matrices associated with a graph are subsets of the real $n \times n$ positive semidefinite matrices, which we denote by $PSD_n$, and the complex $n \times n$ positive semidefinite matrices, which we denote by $HPSD_n$. The *set of symmetric positive semidefinite matrices of graph* $G$ is

$$SD(G) = \{A \in PSD_{|G|} : \ G(A) = G\},$$

and the *set of Hermitian positive semidefinite matrices of graph* $G$ is

$$HSD(G) = \{A \in HPSD_{|G|} : \ G(A) = G\}.$$

Then we define the *minimum semidefinite rank of a graph* $G$, denoted by $msr(G)$, the smallest rank over all matrices in SD(G). It is clear that $mr(G) \leq msr(G)$. Along these lines, we define $M_+(G)$ to be the maximum nullity over all matrices in SD(G). It is evident that $M(G) \geq M_+(G)$, for all graphs $G$.

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During the week at BIRS our team considered a number of important open problems regarding minimum rank and minimum semidefinite rank. One such issue, which is of current interest, was to consider the class of graphs known as outerplanar. A graph is *outerplanar* if it has a crossing-free embedding in the plane such that all vertices lie on the same face. It is worth noting that all trees and all unicyclic graphs are outerplanar.

Associated with any graph is an important graph parameter, known as the path cover number. The *path cover number* of a simple graph $G$, $P(G)$, is the minimum number of vertex disjoint paths occurring as induced subgraphs of $G$ that cover all of the vertices of $G$. It is known that $M(T) = P(T)$ for every (simple) tree $T$ [2]. Sinkovic has recently demonstrated that for a (simple) outerplanar graph $G$, $M(G) \leq P(G)$ and has given a family of outerplanar graphs for which equality holds [5].

One of our main objectives for the week was to gain a better understanding on the possible connections between the minimum rank and minimum semidefinite rank of a graph, and this is exactly what we accomplished in the case of outerplanar graphs. We began by discovering a new graph parameter, known as the tree cover number and used it in connection with $M_+(G)$ when $G$ is outerplanar.

The *tree cover number* of a graph $G$, possibly with multiple edges but no loops, denoted $T(G)$, is the minimum number of vertex disjoint simple trees occurring as induced subgraphs of $G$ that cover all of the vertices of $G$.

Our main result is a complete characterization of the maximum nullity over all positive semidefinite matrices whose graph is outerplanar. Namely, we proved that $M_+(G) = T(G)$, for all outerplanar graphs $G$. This is a significant result, as like the case of trees, it establishes, a direct link between the algebraic quantity, nullity, to a combinatorial quantity, namely the tree cover number. Moreover, this result verifies an equation between $msr$ and a graph parameter. The main tool used in the proof of this theorem is the notion of orthogonal removal of a vertex, which was developed in the context of finding the minimum semidefinite rank of chordal multigraphs [1].

We also studied the tree cover number in general, and compared with other known graph parameters and to $M$ as a completeness exercise. We were also faced with a number of interesting open questions, such as studying the graph complement conjecture for outerplanar graphs, along with many other issues.

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Given a set $\mathbf{X}$ of $n$ nonzero column vectors in $\mathbf{C}^d$, $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, let $X$ be the matrix $[\ \mathbf{x}_1 \quad \ldots \quad \mathbf{x}_n\ ]$. Then $X^*X$ is a psd matrix called the *Gram matrix* of $\mathbf{X}$ with regard to the Euclidean inner product. Its associated graph $G$ has $n$ vertices $\{v_1, \ldots, v_n\}$ corresponding to the vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, and edges corresponding to nonzero inner products among those vectors. By the *rank* of $\mathbf{X}$, we mean the dimension of the span of the vectors in $\mathbf{X}$, which is equal to the rank of $X^*X$. Consequently, $\mathbf{X}$ is called a *vector representation* of $G$. Vector representations have been a key tool in recent advances in minimum semidefinite rank problems (see, for example, [2, 3]).

At BIRS, it was shown that vector representations can be used in conjunction with unitary matrices to solve or give new approaches to open problems:

Given a vector representation of a graph $G$, let $X$ be the matrix mentioned above. Let

$$P = \begin{bmatrix} X \\ E \end{bmatrix},$$

where $E$ is a matrix whose zero/nonzero pattern is that of the edge-vertex incidence matrix of $G$. We first notice that, since the columns of $X$ give a vector representation of $G$, and since the rows of $E$ have only two nonzero entries each corresponding to a nonzero inner product of columns of $X$, the nonzero entries of $E$ can be specified so that the columns of $P$ are pairwise orthogonal. After normalizing the columns of $P$, $P$ may be completed to a unitary matrix

$$V = \begin{bmatrix} X & ? \\ E & L \end{bmatrix},$$

where the rows of $L$ must then be a vector representation of the line graph of $G$, $L(G)$. Inspecting the sizes of the various blocks of $V$, this gives the following result, obtained at BIRS this year: For any graph $G$, $|G| - \mathrm{msr}(G) \leq |L(G)| - \mathrm{msr}(L(G))$.

Written slightly differently as

$$\mathrm{msr}(L(G)) \leq |L(G)| - (|G| - \mathrm{msr}(G)),$$

this is reminiscent of the $\delta$-conjecture, one of the two most well-known open conjectures in minimum rank: For any graph $G$, $\mathrm{msr}(G) \leq |G| - \delta(G)$, where $\delta(G)$ is the smallest degree among the vertices of $G$. Significant progress was made in special cases of the $\delta$-conjecture at BIRS: including a proof of the conjecture when $\delta \leq 3$. Further, it was conjectured that a stronger result is true, namely, that $M_+(G) \geq \tilde{\delta}$, where $\tilde{\delta}$ is the maximum degree of a vertex $v$ that has minimum degree among its neighbors, and $D(v)$ (i.e., the graph obtained from $G$ by deleting $v$ and all of the neighbors of $v$) is connected.

Another well-known open conjecture is the *Graph Complement Conjecture*: For any graph $G$, if $\overline{G}$ is the complement of $G$, then $\operatorname{msr}(G) + \operatorname{msr}(\overline{G}) \leq |G| + 2$.

As noted at BIRS, both the graph complement conjecture (GCC) and the $\delta$-conjecture can be transformed into associated unitary matrix completion problems. This idea is similar to one previously explored in the context of finding the msr of bipartite graphs [4]. Here, we will demonstrate how to approach GCC:

Let $X$ be a matrix whose columns form a minimal vector representation of $G$. Construct a matrix $E$ whose rows have exactly two nonzero entries, and where each row of $E$ corresponds to either an edge of $G$ or an edge of $\overline{G}$. Thus $E$ will be a $\left( \begin{array}{c} |G| \\ 2 \end{array} \right) \times |G|$ matrix. Let

$$M = \left[ \begin{array}{c} X \\ E \end{array} \right].$$

Choose the nonzero entries of $E$, row by row, so that if the columns of $X$ corresponding to the two nonzero entries of a row in $E$ are not orthogonal, then the corresponding columns of $M$ are, and vice-versa. At the end of this process, the columns of $M$ will be a vector representation of $\overline{G}$, but most likely not a useful one, as it will no doubt have a high rank. Now, find a matrix $N$ so that the rows of the matrix $\left[ \begin{array}{cc} M & N \end{array} \right]$ are pairwise orthogonal and all have the same length (we discuss how to do this below). Having done so, normalize the rows of $\left[ \begin{array}{cc} M & N \end{array} \right]$, and extend to a unitary matrix

$$U = \left[ \begin{array}{cc} M & N \\ V & ? \end{array} \right].$$

By construction, the columns of $V$ give a vector representation of $\overline{G}$ with rank bounded by a function of the size of $N$. In particular, if such an $N$ can be selected to have $\left( \begin{array}{c} |G| \\ 2 \end{array} \right) + 2$ columns, then

$$\operatorname{msr}(\overline{G}) \leq |G| + 2 - \operatorname{msr}(G),$$

establishing GCC. If any such $N$ must have more than $\left( \begin{array}{c} |G| \\ 2 \end{array} \right) + 2$ columns for a particular graph, then that graph will give a counterexample for GCC.

We note that such an $N$ may always be found, as the question of simultaneously normalizing and orthogonalizing a set of vectors can be phrased as the matrix equation

$$M^*M + N^*N = cI,$$

where $M$ is known. Since $M^*M$ is a positive semidefinite matrix, choosing any $c > \max \sigma(M^*M)$ (where $\sigma$ is the set of eigenvalues) will make $cI - M^*M$ a positive semidefinite matrix, and guarantee the existence of an $N$ with $N^*N = cI - M^*M$.

We can phrase this question, then, in a number of different but equivalent ways: Given a zero/nonzero pattern, what is the size of the smallest pattern containing the original that is the pattern of a unitary matrix? What is the largest multiplicity of the largest eigenvalue of a Hermitian matrix with given zero/nonzero pattern? What is the smallest rank matrix $N$ that will solve the matrix equation $M^*M + N^*N = cI$ for given $M$ and arbitrary $c$?

# References

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