# GLOBAL DYNAMICS OF STOCHASTIC DIFFERENTIAL DELAY EQUATIONS

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## 1 Overview of the Field and Some Open Problems

Stochastic functional differential equations represent a relatively new field of the qualitative theory of differential equations. Their significance has become more evident in recent years due to a great variety of their applications in modeling real life phenomena. Delays are intrinsic features in a multitude of processes in applied sciences and engineering. Uncertainty of the available data and/or the randomness of aspects of the processes themselves lead to the presence of random elements in the models, thus resulting in stochastic differential delay equations. Though the theory of both deterministic functional differential equations and the stochastic ordinary differential equations are rather well developed areas of research, the qualitative theory of stochastic differential delay equations is largely in its infancy stage. Partial explanation of such state of things is the sometimes enormous difficulties facing the researchers, in their approaches and attempts to solve even simply formulated problems. For example, conditions for the stability of the following simple linear stochastic differential delay equation with constant coefficients

$$dx = (ax(t) + bx(t-\tau))\,dt + (\sigma_0 x(t) + \sigma_1 x(t-\tau))\,dW(t) \tag{1}$$

are not derived yet. Many aspects of the basic theory of stochastic functional differential equations still need to be developed. Note that the book [5] contains most of the basic theory presently available for stochastic functional differential equations. See also works [1, 3, 7] for additional related details and open problems.

Our intention as a group is to approach some of the problems in the field from a unified point of view, as small stochastic perturbations of some well known deterministic processes. From this prospective, and as a part of the program of our RiT Workshop at the BIRS, we are trying to study the effects of stochastic elements on one-dimensional dynamical systems and continuous time difference equations [4, 6].

## 2 Scientific Progress Made

During the meeting we have discussed a number of models appearing in recent applications that are described by stochastic functional differential equations. Among others, the models include equations frequently used in finance applications, such as the geometric Brownian motion, the Ornstein-Uhlenbeck process, the Vasicek process, and the continuous GARCH process. We have developed a unified approach to tackle a range of problems related to the above processes and a detailed plan of their future studies. In particular, we are aiming at solving a range of control problems for the general stochastic equation with delay

$$dS(t) = a(S_t, S(t), u(t))dt + b(S_t, S(t), u(t))dW(t), \tag{2}$$

where one is looking to optimize a certain related functional (such as a cost functional, a consumption functional, etc.). Some of the ideas to be used and further developed are based on our recent paper [2].

We have also looked at several specific problems of global dynamics in the stochastic differential delay equation

$$dx(t) = [f(x(t-\tau)) - a\,x(t)]dt + g(x_\tau)dW(t). \tag{3}$$

The problems include the global stability of a unique steady state, instability and bifurcation of equilibria, existence of periodic solutions and their stability and shape, and dependence of solutions on parameters. Some of those problems have been stated and partial solutions derived for some of the respective deterministic equations. To the best of our knowledge, those questions are not addressed yet for the stochastic equation (3). We have achieved a good progress in solving several of those problems, in particular by treating equation (3) as a perturbation of the limiting difference equation $x(t) = 1/a\ f(x(t-\tau))$. This work should result in a joint publication (to be submitted soon). We have also developed a plan of further joint studies in this direction.

# 3 Outcome of the Meeting

The purpose of the one-week meeting at the BIRS has been two-fold. The first one was to develop a program of joint research in particular directions of stochastic differential delay equations that are in the intersection of mutual interests of the participants. We have achieved this goal by identifying a number of applied models with the related equations that we will approach to study various aspects of their dynamical behavior.

The second part of the main objective was to further advance and to complete several aspects of joint ongoing research that have been in the working lately between the participants. We have succeeded in this part too. In particular, Ivanov and Swishchuk have completed a typescript dealing with the problem of global stability in a stochastic differential delay equation which is a singular and random perturbation of a continuous time difference equation. Ivanov and Khusainov have completed their work on certain representations of solutions for partial differential equations with delay. Both works have been submitted for publication. Two more manuscripts are near completion.

# References


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